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Analysis and synthesis of linear electrical networks by graphical means

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ANALYSIS AND SYNTHESIS OF LINEAR ELECTRICAL NETWORKS BY GRAPHICAL MEANS

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ANALYSIS AND SYNTHESIS OF LINEAR ELECTRICAL NETWORKS

BY GRAPHICAL MEANS

by

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ABSTRACT

This thesis, the result of a literature search conducted at the Naval Postgraduate School, is a collection of different geometrical methods and their applications in respect to network analysis and synthesis in electrical engineering.

Part one deals primarily with finding the equivalent of two or more impedances in parallel. The second part is directed towards the application of these methods to the problem of matching two arbitrary impedances in order to obtain maximum power transfer.

A comprehensive bibliography is included.

Thesis by: Detlef Bruno Kammholz and Günther Ferdinand Pirschl entitled Analysis and Synthesis of Linear electrical Networks by Graphical Means.

ERRATA

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INTRODUCTION

The application of mathematics to an engineering problem always requires a numerical calculation in order to find one or more unknowns which are related to the given parameters. This computation, especially if complex numbers are involved, very often becomes quite laborious. It was, therefore, in the interest of many to find a simple, fast, and sufficiently accurate way to obtain the needed numerical answers. The graphical method comes very close to fulfilling these requirements and it gains special importance if used by the engineer as a means to reach rapidly a desired design goal, although ultimate accuracy depends upon the utilization of additional analytical methods - a method which allows the crosschecking of both results. Additional advantages of the graphical method to be noted are, first, the possibility that one can easily follow the path leading to the final solution and thereby can gain more insight into the problem; and second, the ability of one to better observe the interdependence of the given parameters on each other.

Graphical techniques - so often missing in modern textbooks - can be an excellent tool to enhance the basic understanding of analytic procedures. Errors can be avoided more easily because many methods are straightforward and lead directly to a solution.

The fundamental work on graphical methods was done by M. d'Ocagne of Paris more than 50 years ago, and the publication of one of the earliest papers by the German, R. Neumann, giving the graphical solution for finding the total impedance of a Wheatstone bridge, was traced back to the year 1914. From then on an extensive number of papers has been produced, many presenting new ideas and methods; many, however, just altering slightly or even duplicating previous work done.

The many different methods of the field treated in this thesis can be put into a rather simple order upon which the partition of this paper will be based.

In Part I methods for network analysis are developed, with the main objective of finding graphically the equivalent impedance of two or more impedances in parallel. First, the simpler, special cases of paralleling like impedances are treated, progressing then to a discussion of the more difficult cases of paralleling two impedances separated by a phase angle of 180 degrees and 90 degrees, and finally to end with a description of the general cases of paralleling two impedances separated by any phase angle. Part I is concluded with two methods of finding the equivalent impedance of a bridge network composed of like impedances only.

In Part II methods for network synthesis are developed, the main objective being to find simple matching networks so as to insure maximum power transfer between two impedances.

PART I: GRAPHICAL METHODS IN NETWORK ANALYSIS

1. The Series Parallel Conversion.

In one of the early papers Ohrlich introduced a simple method to find the equivalent parallel parameters R_p and X_p from the given series parameters R_s and X_s of a given impedance Z, and vice versa. As shown in Figure 1-1 R_p and X_p can be found by drawing a perpendicular through the tip of Z and letting this perpendicular intersect with the x- and y-axis, this giving points B and C respectively. The phasors OB and OC are then R_p and X_p respectively. The input impedances of the series and parallel circuits shown in Figure 1-2 are identical. The proof is presented in Appendix 1. The application of this impedance triangle in the field of graphical analysis and synthesis is widespread, as the reader will see in this paper.

One useful application lies in the possibility to find the equivalent impedance of a resistance R_p in parallel with a reactance X_p. This can be done by doing the above described method "backwards."

2. The Paralleling of Two Like Impedances.

Among the engineers who have worked in the field of graphical analysis, one of the best known is probably Rukop who with his early work in this field stimulated a great number of engineers to look further into the subject.

The following method, worked out in Figure 2-1, is credited to Rukop and is likely to be the most general method to find the equivalent impedance of two like impedances in parallel. Like impedances are impedances that have the same phase angle; two resistances, for example, are two like impedances.

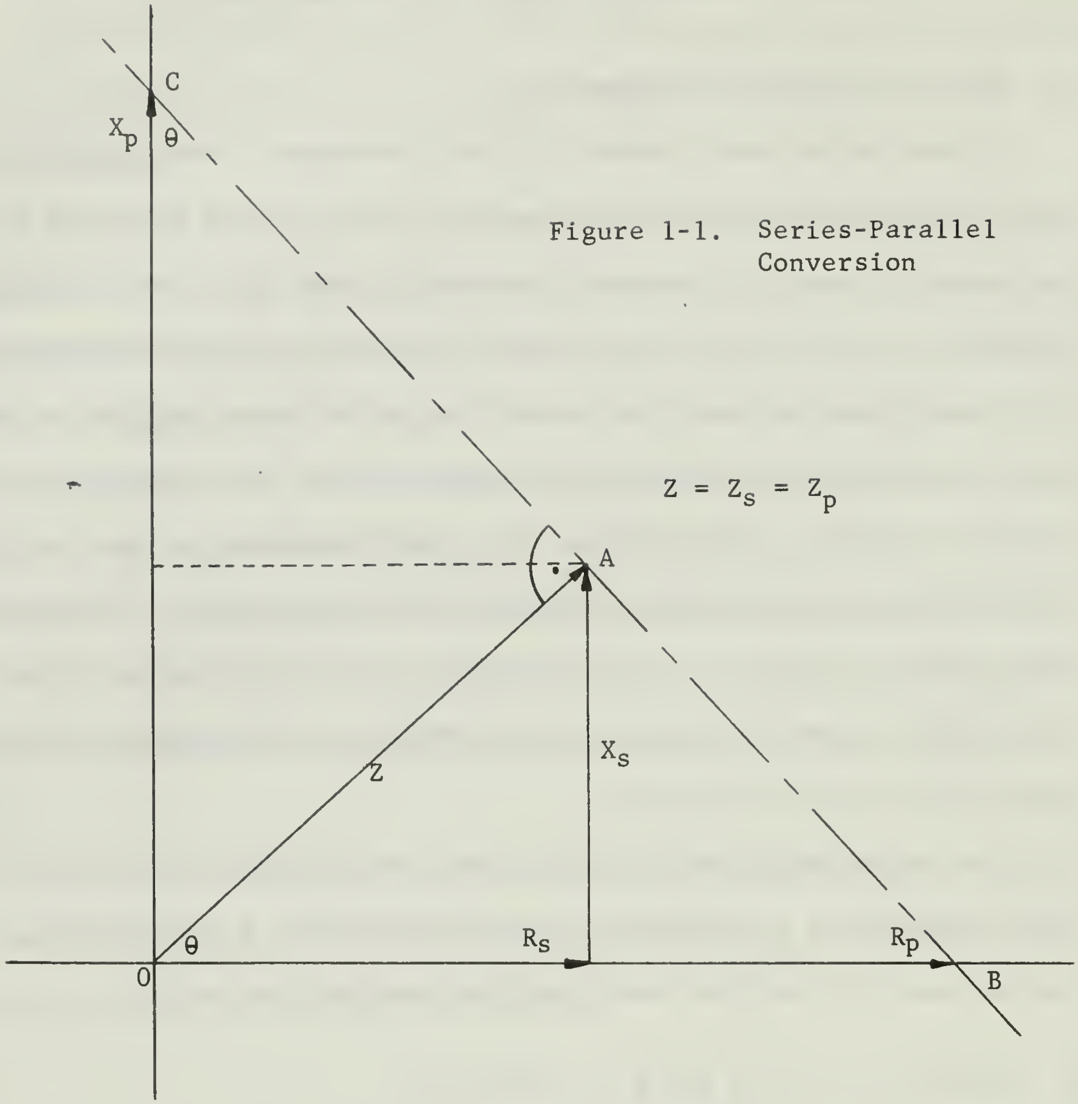

Figure 1-1. Series-Parallel Conversion

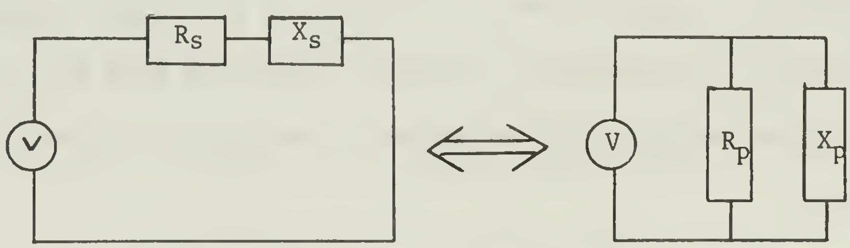

Figure 1-2. Series and Parallel Circuits, Representing Identical Input Impedances

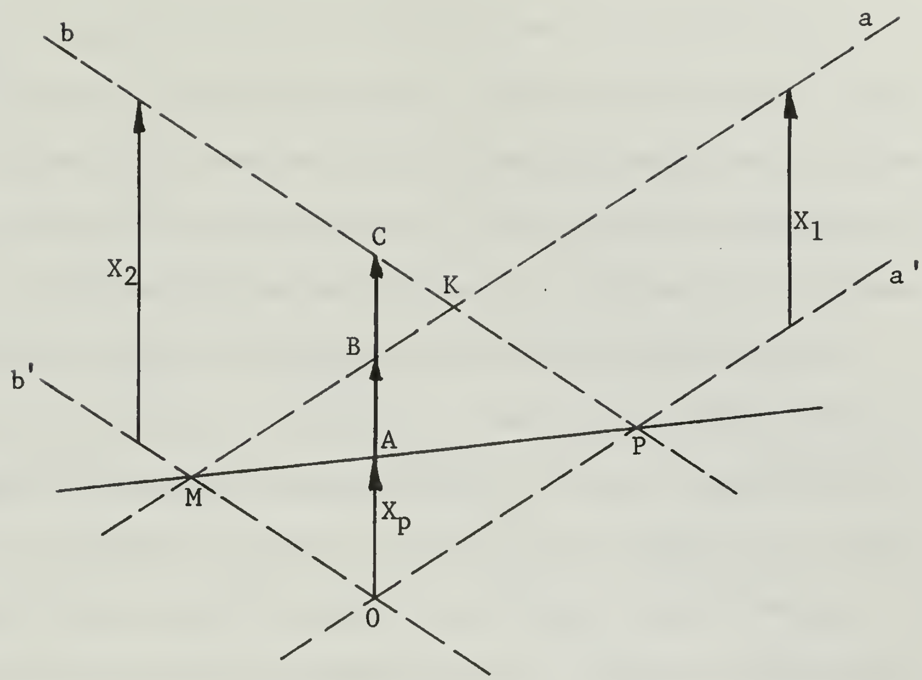

Figure 2-1. Paralleling of Like Impedances, Rukop's Method

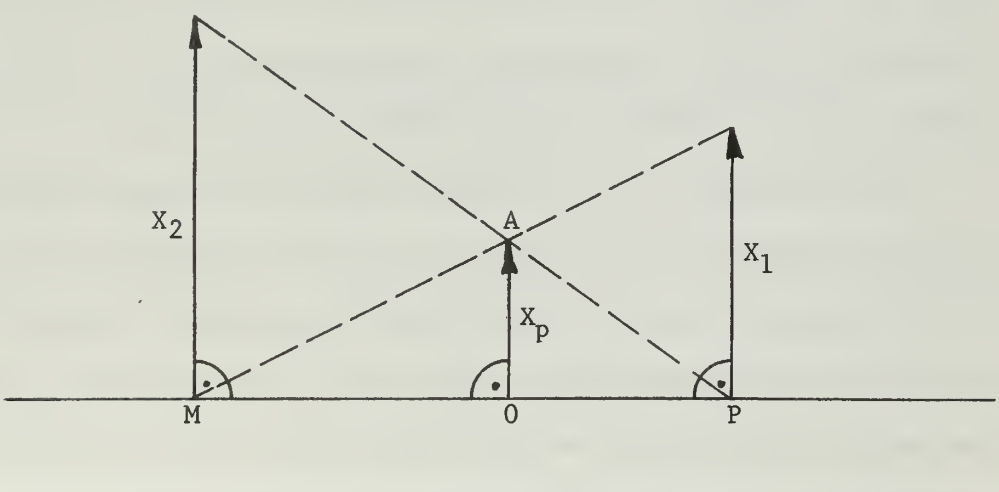

Figure 2-2. Paralleling of Like Impedances, Reducing Construction by Using Same Baseline

In Figure 2-1 the two vectors X_1 and X_2 represent two like impedances. Two sets of parallel, intersecting lines (any two sets) a, a' and b, b' are drawn through the tips and ends of X_1 and X_2. Through the intersections of a with b' and of b with a' line MP is drawn. OA, parallel to X_1 and X_2, is then the desired equivalent X_p.

In Figure 2-2 the construction becomes simpler by drawing X_1 and X_2 perpendicular to the same base line MP. X_1 and X_2 can be separated by any distance. X_p, the desired equivalent impedance, can be found easily. The proof of this method is given in Appendix 2.

In 1943 Paine, another more popular figure in the field of graphical methods, published a simple method to parallel two like impedances. In Figure 2-3 R_1 and R_2 are laid out on the x-axis, represented by the phasors OA and AB respectively. Somewhere on the y-axis a parallel to the x-axis is drawn. FD is made equal to R_2 and the extension of the line BD cuts the y-axis in point E. The intersection of AE with FD gives point C and FC then represents the parallel value R_p. The proof that $R_p = \frac{R_1 R_2}{R_1 + R_2}$ is easy because similar triangles yield the following relations: $FC/FD = OA/OB$ which leads directly to $FC = FD\,\frac{OA}{OB}$, the desired relation, because $FC = R_p$, $OA = R_1$ and $AB = R_2$.

R. D. Douglass and D. P. Adams present one further method to parallel two like impedances. In Figure 2-4 X_1 and X_2 are laid out on rectangular axes and form the right triangle OAB. The parallel value X_p is given by the sides of the maximum square that can be drawn into this right triangle. The median axis OC helps to find this maximum square, because point C is one corner of the square. Proper scaling of the median axis creates a nomograph for the resultant impedance X_p.

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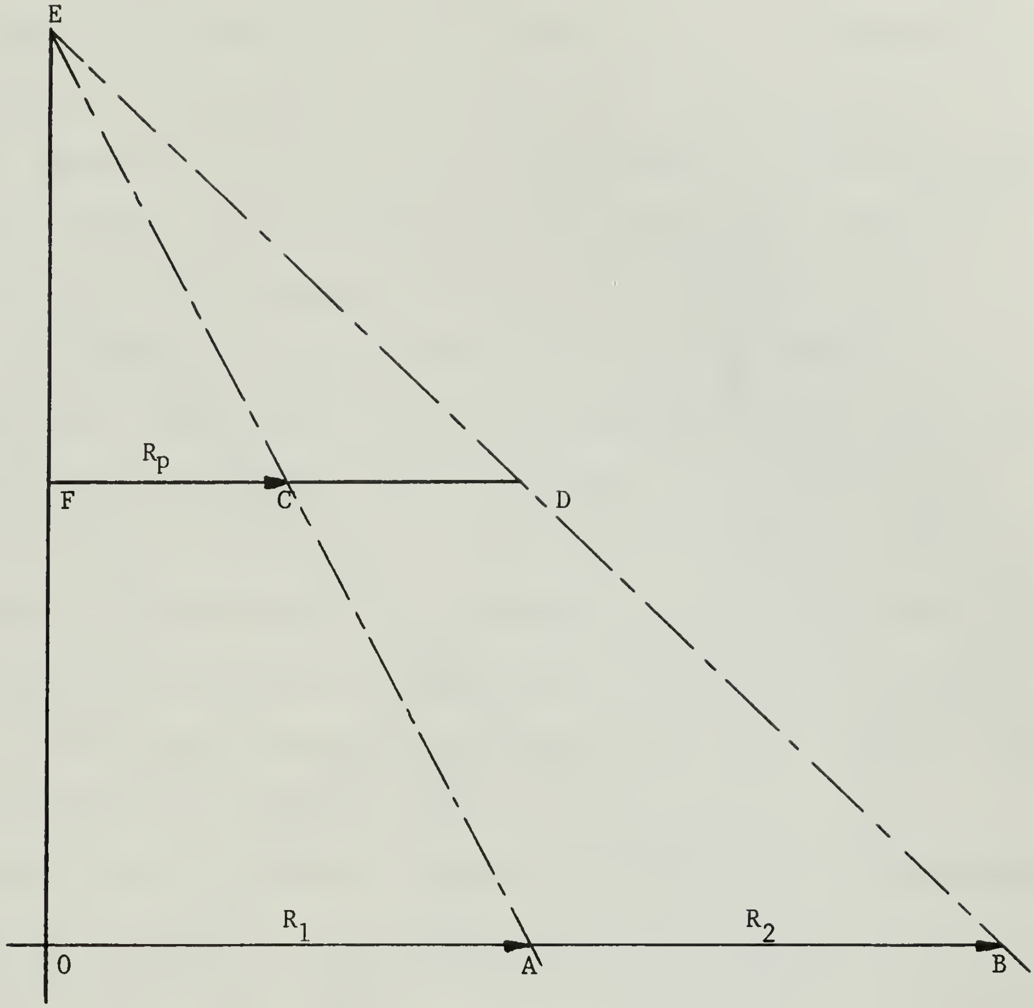


Figure 2-3. Paralleling of Like Impedances, Simple Method by Paine

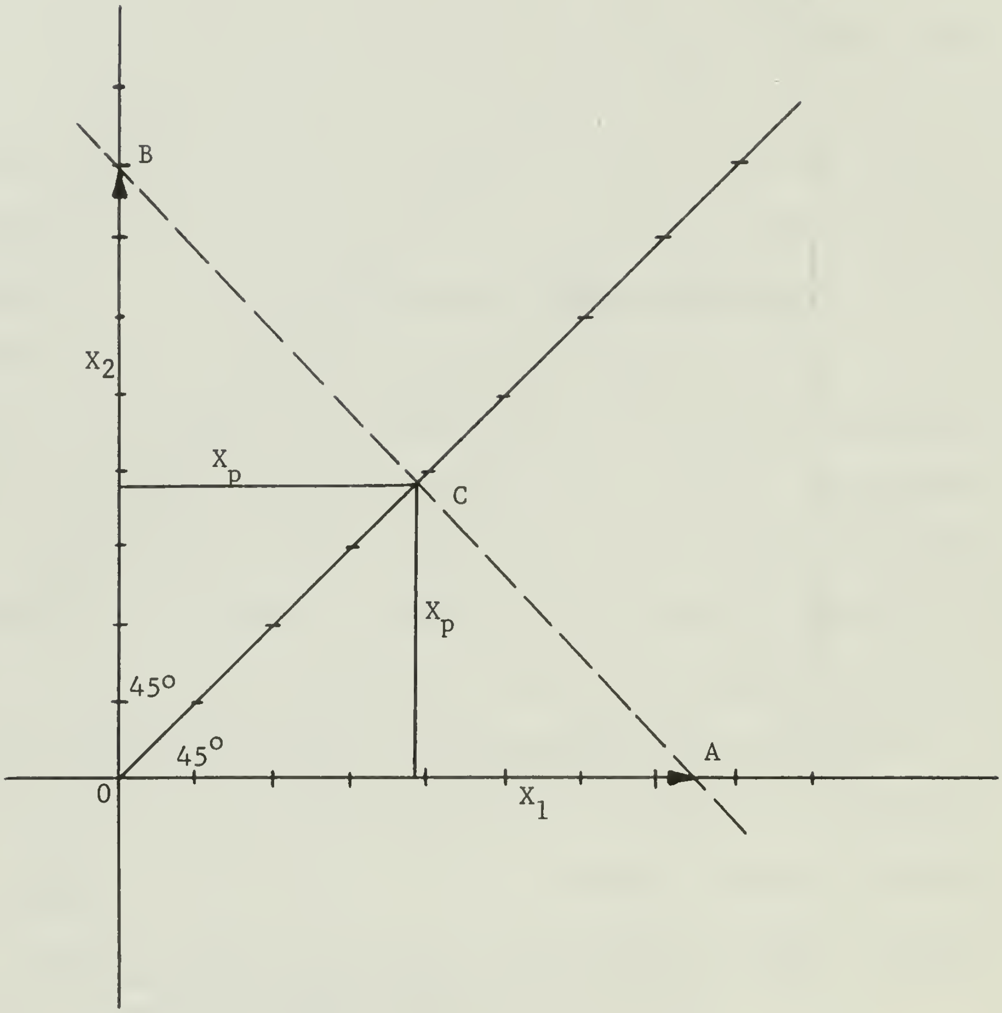


Figure 2-4. Paralleling of Like Impedances, Using Maximum Square

method depicted in Figure 2-5. X_1 and X_2 are laid out on axes 120 degrees apart. The intersection of the median axis with the line AB, that connects the two vector tips, gives point C. OC is then the parallel impedance X_p. As can be seen from Figure 2-6, this method can be used to parallel two impedances with 180 degrees phase difference. The proof is given in Appendix 3 in connection with Figure 2-7.

The above method can be extended easily to the configuration shown in Figure 2-8. This method, developed by Meyer zur Capellen, uses nomography as did the method depicted in Figure 2-4. Six starlike axes are drawn all 60 degrees apart and each spike has the same scale. With this nomograph an unlimited number of like impedances can be paralleled. In this example five impedances in parallel give the equivalent impedance Z_{p4}. Z_1 and Z_2 yield the equivalent impedance Z_{p1}, which itself is then paralleled with Z_3 to yield Z_{p2}, and so on.

Storch, too, has developed a method to parallel two like impedances, as shown in Figure 2-9. A reference line in an arbitrary direction through point 0 is drawn, here XOY. Z_1 and Z_2 are laid off in the same direction and two circles, both centered at 0, are drawn through their tips. Rhombus OXAB has to be formed from the already existing sides OX and OB, thus having all sides equal in magnitude to $|Z_1|$. Finally Y and A are connected by a straight line and the resulting phasor OD on line OB represents the desired parallel impedance Z_p. In Figure 2-9 the similar triangles OYD and YXA can be spotted right away and the relations derived from them lead to a simple proof: OD/OY = AX/XY which gives at once the result $Z_p = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$ as OD = Z_p, OY = $|Z_2|$, AX = $|Z_1|$ and XY = $|Z_1| + |Z_2|$.

In one of the early papers Kind demonstrates a method in which four

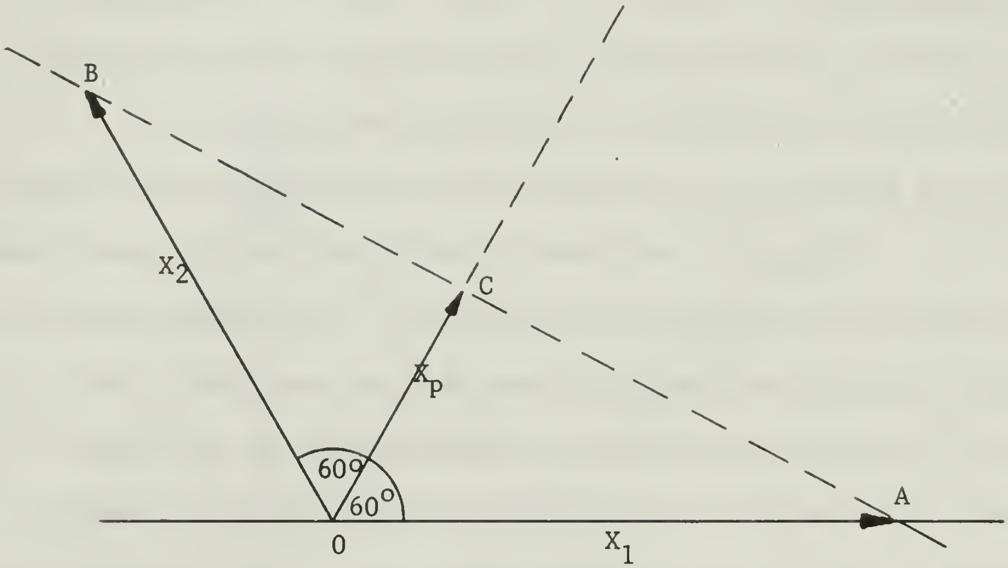


Figure 2-5. Paralleling of Like Impedances, Axes 120° Apart

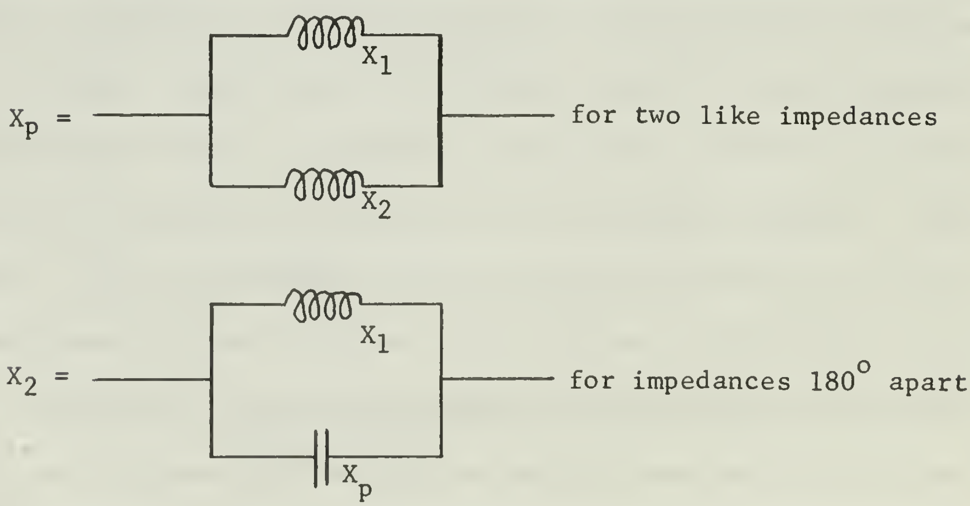


Figure 2-6. Two Different Applications for Method in Figure 2-5

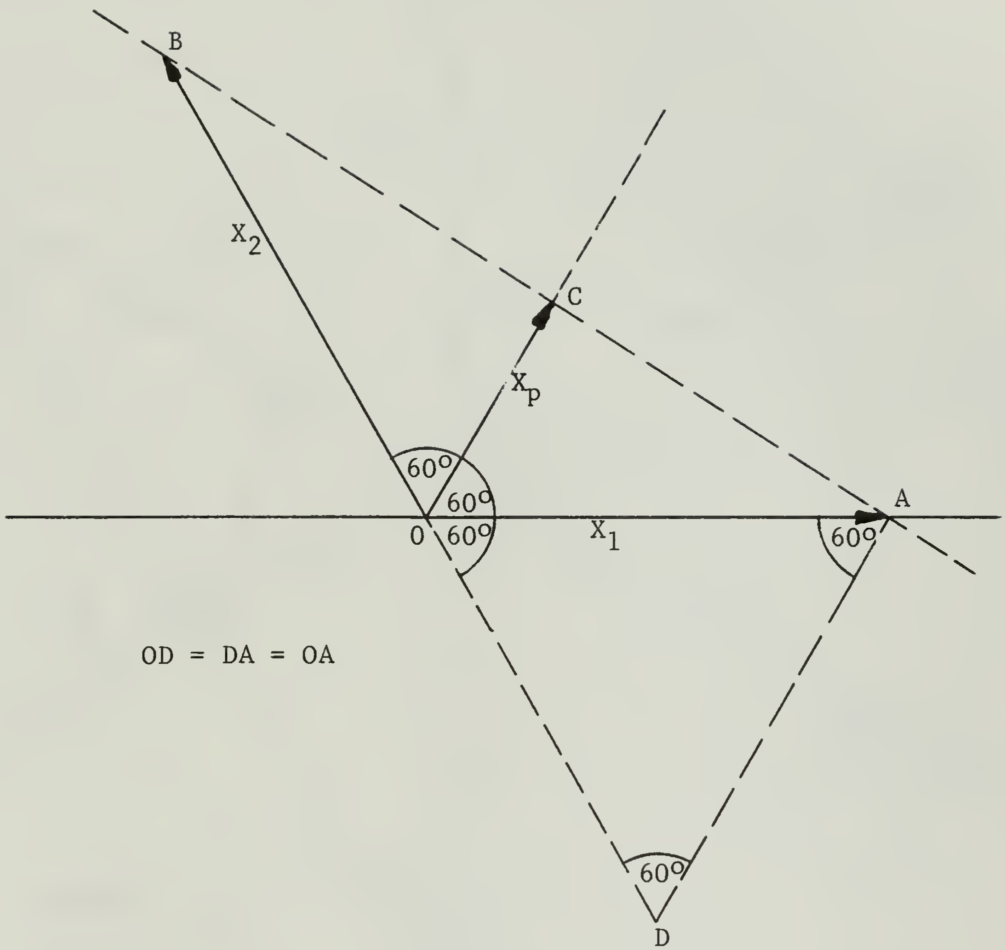


Figure 2-7. Paralleling of Like Impedances, Axes 120° Apart, With Similar Triangles for Proof in Appendix 3

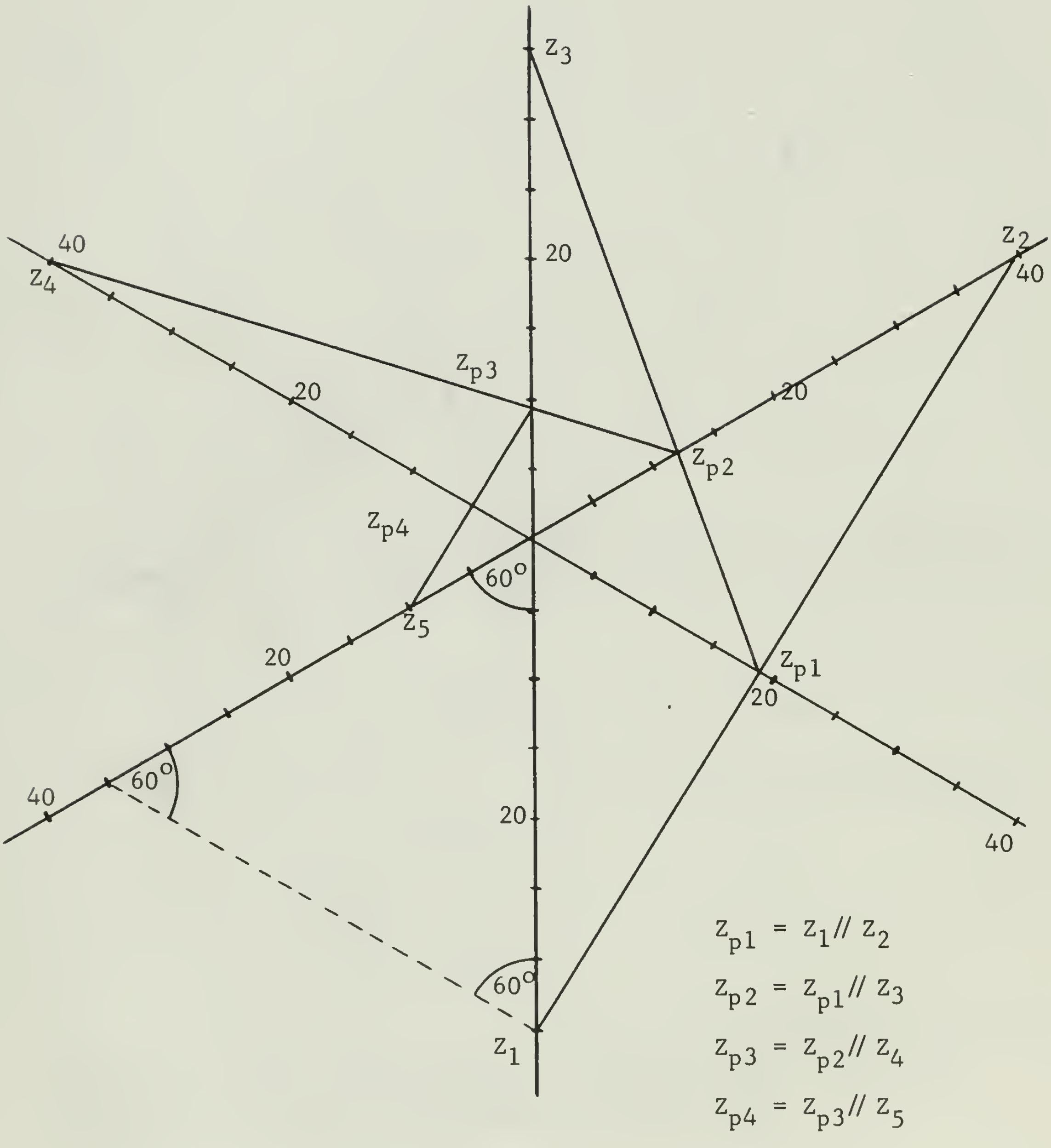


Figure 2-8. Example of Application of Method Shown in Figure 2-5

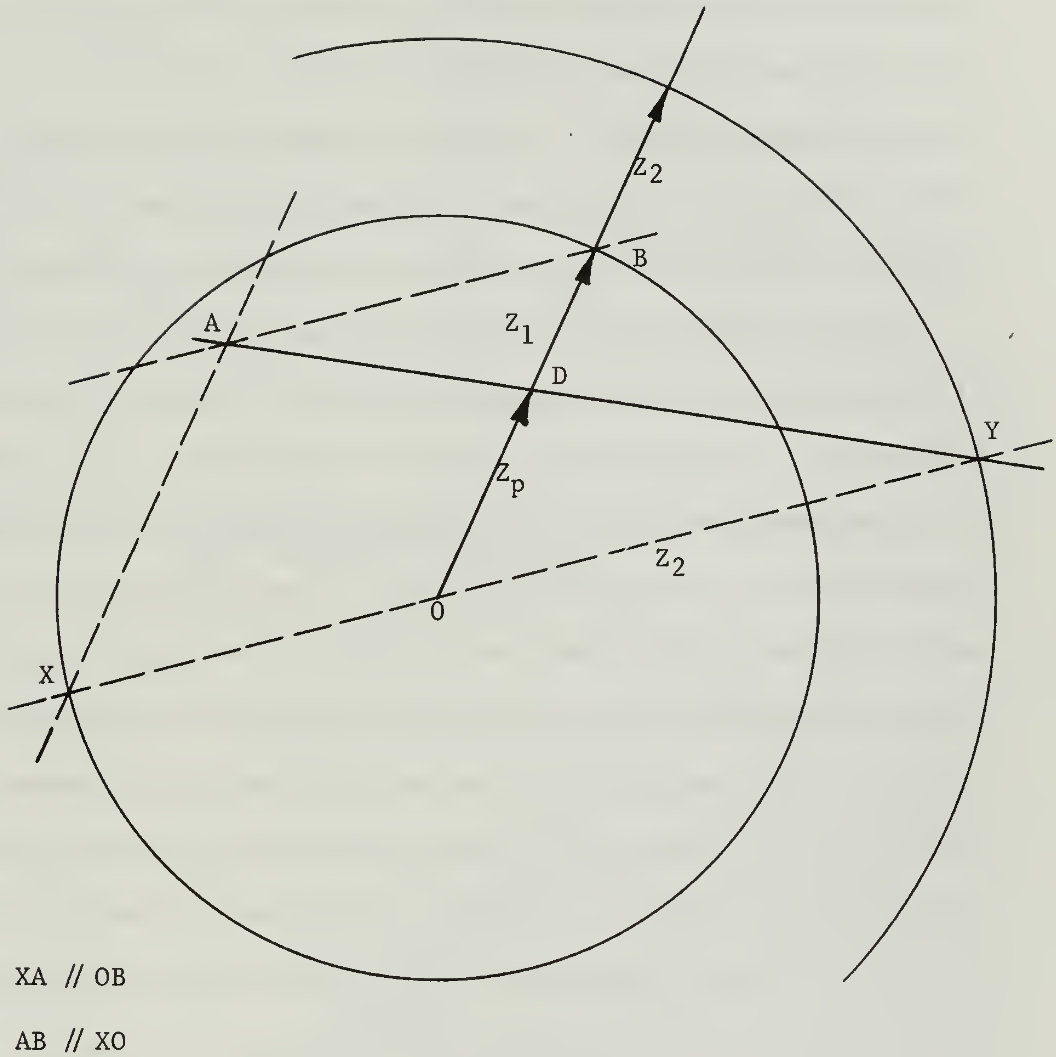


Figure 2-9. Paralleling of Like Impedances, Method by Storch

arcs are sufficient to find the parallel equivalent of two like impedances. In Figure 2-10 Z_1 and Z_2 are the given impedance vectors. Arcs are drawn through origin and tip of each vector with radii equal to the corresponding vector lengths. Hereby care should be taken that one of the arc-triangles should be placed to the right, the other one to the left of the impedance vectors. A straight connection between their intersection points A and B then gives the parallel equivalent Z_p on their mutual base line. The proof of this method is presented in Appendix 4.

Boening, whose name appears more often in connection with this field of electrical engineering, proposed a further method to parallel two like impedances. This method is pictured in Figure 2-11. The reciprocals of the given impedances Z_1 and Z_2 are added linearly. In this case the reciprocal of Z_1 is Z_2 and vice versa, can be found by applying the secant and tangent theorems simultaneously. The tangent theorem states that a tangent is the mean proportional between a secant and its external segment if the tangent and the secant are drawn from a common point outside the circle. The secant theorem states that for two secants drawn from an external point of a circle, the products of the whole secants with their external segments are equal. The circle is drawn with arbitrary radius, but it must pass through the tips of Z_1 and Z_2. Since OA = OB, the latter can be inverted to give OD, the parallel equivalent impedance in magnitude. OD measured off the common base line of Z_1 and Z_2 yields Z_p in magnitude and phase.

3. The paralleling of Two Impedances 180 Degrees Out of Phase.

After having dealt until now with methods to solve parallel networks with like impedances, the next step will be to look at the problem of paralleling two impedances 180 degrees out of phase, as is the case when

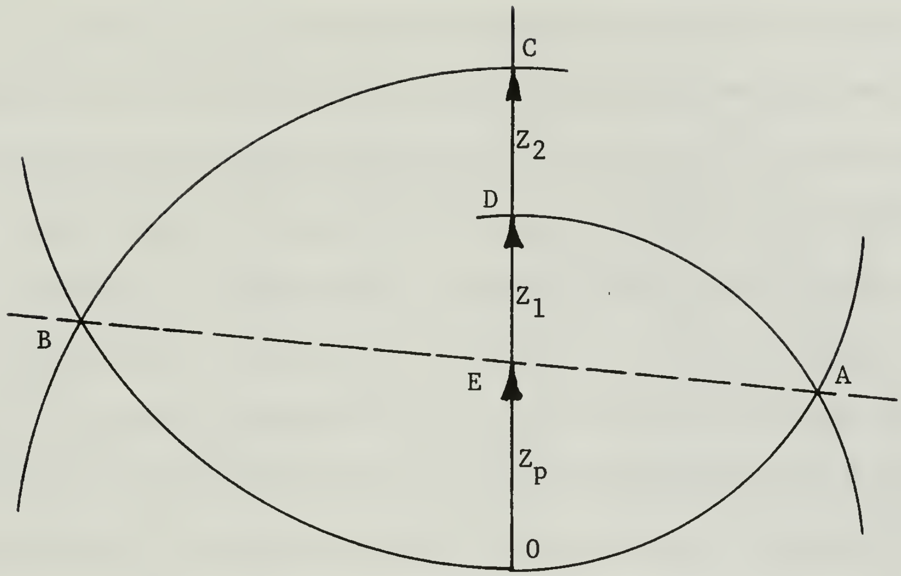


Figure 2-10. Paralleling of Like Impedances, Resorting to Four Arcs for Construction

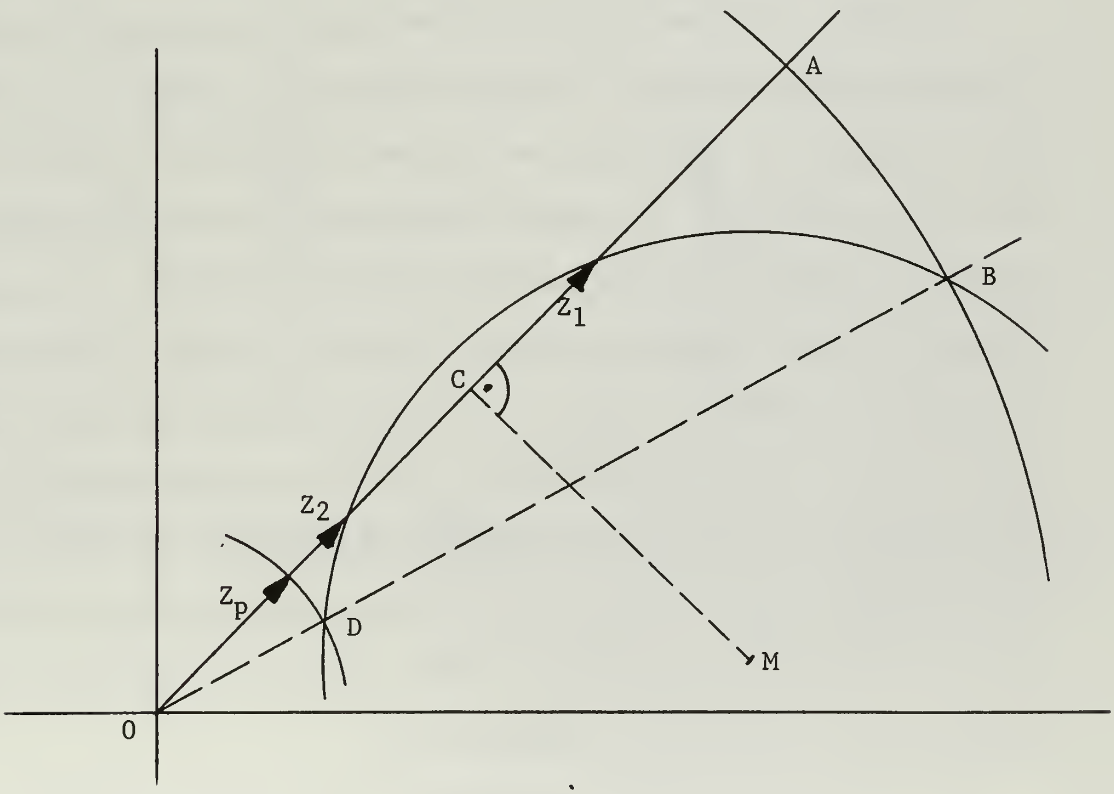


Figure 2-11. Paralleling of Like Impedances, Applying Secant and Tangent Theorems

an inductance and capacitance are connected in parallel.

The method shown in Figure 3-1 originated with Wilkinson. Given are Z_1 and Z_2 along a reference line, here the y-axis. The smaller impedance, here Z_1, is laid off in its opposite direction giving OD. In D a perpendicular to the y-axis DC is erected, in length equal to $|Z_1|$. E, the tip of Z_2 is connected with C. Then point B is found by forming a right angle at C with line EC. To find point F, the distance DB is measured along the y-axis from D in its opposite direction. OF then represents the parallel equivalent in magnitude and phase Z_p. The proof is easily deducted from the theorem that states that the perpendicular drawn from the right angle onto the hypotenuse in a rectangular triangle, is the mean proportional between both segments on the hypotenuse. This statement is proven in Appendix 5.

In Figure 3-2 a method is described that was elaborated by Boehne. It uses the proportionality theorems of tangents, secants, and chords (see Appendix 15). The secant and tangent theorems are stated in connection with Figure 2-11. The chord theorem says that for two intersecting chords the products of their segments are equal. First a circle with arbitrary radius through the tips of Z_1 and Z_2 is drawn. The magnitude of the smaller impedance, here Z_1, is subtracted from the larger one, giving point A. With its origin at 0 a circle is drawn through A; this circle cuts the original circle in point B, yielding $OB = 1/Z_p$. Line BC represents a secant and Z_p can be found by getting point D, the intersection of the circle centered at 0 and of radius OC with the y-axis. OD represents the equivalent impedance Z_p.

D. W. Wells introduced an extension of the method depicted in Figure 2-5 to impedances 180 degrees in phase apart. The two impedances

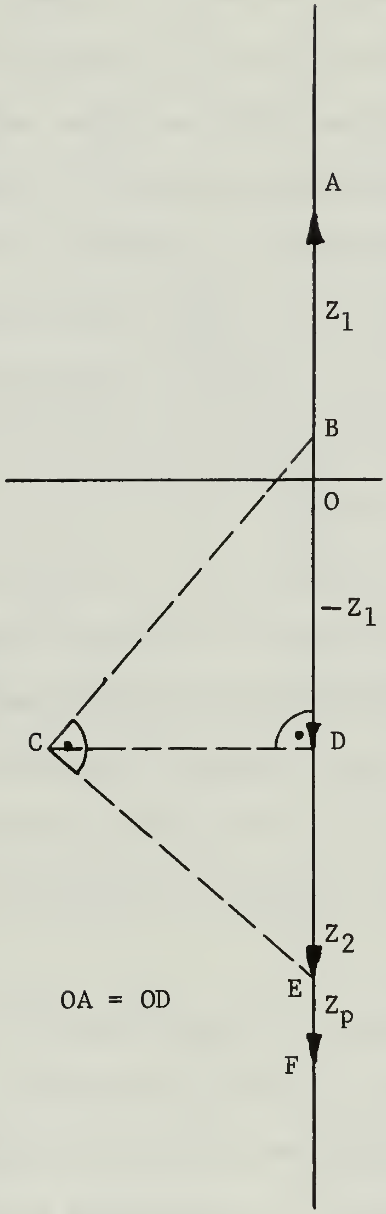


Figure 3-1. Impedances 180° Apart, Method by Wilkinson

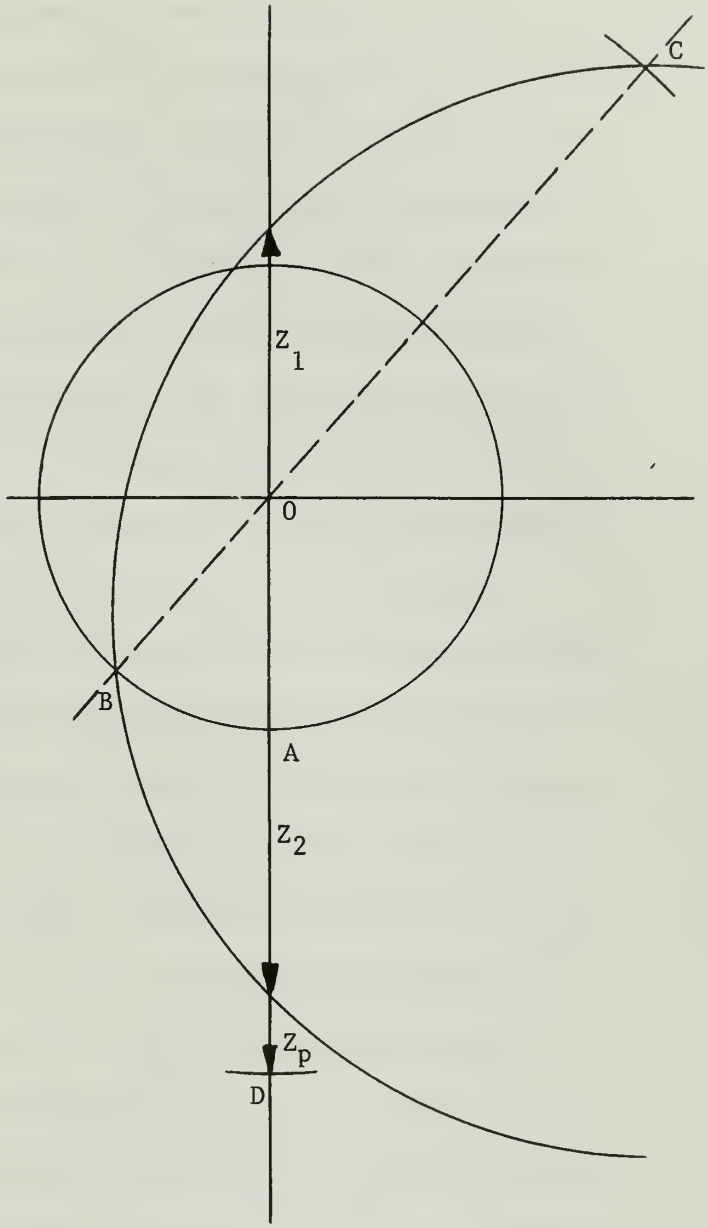


Figure 3-2. Impedances 180° Apart, Applying Tangent, Chord, and Secant Theorems

are laid out along two axes 120 degrees apart, as shown in Figure 3-3. One of the two impedances has to be measured off in the opposite direction as for the case where both impedances have the same phase angle. For example, if X_1' is laid out along the negative x-axis, OA' and OD give the parallel impedance X_p'. The direction of X_p' is that of the "closest" impedance vector, in this case $X_1' = -X_1$. X_2 could as well have been laid out in the direction OB' $= -X_2$ and this time OE represents the equivalent impedance X_p'', where the direction of X_p'' is again that of the "closest" impedance vector, then X_1.

Kind developed another method shown in Figure 3-4. A circle is drawn centered at 0 and passing through the tip of the smaller vector, here Z_1, and another one centered at D, the tip of the larger vector, passing through 0. The common tangent to both circles intersects the x-axis, which serves as reference axis, at A. The equivalent impedance Z_p is then represented by OA. The proof of the above developed method is carried out in Appendix 6.

Boehne describes a method for paralleling two impedances 180 degrees out of phase. In Figure 3-5 an inductance, X_1, and a capacitance, X_c, are laid off on a common reference axis, here the y-axis, with proper phase. The larger impedance, in this case X_1, is transferred to the other side of the x-axis, separated from X_c by a base line OK of arbitrary length. Line AO is drawn and point D determined as intersection of AO with a parallel to the x-axis going through B. The extension of KD cuts the y-axis at point C and OC represents X_p, the desired equivalent value, in magnitude and phase.

Figure 3-5 can be widened to a method in which a fixed, instead of an arbitrary, base line is used as shown in Figures 3-6 and 3-7. In

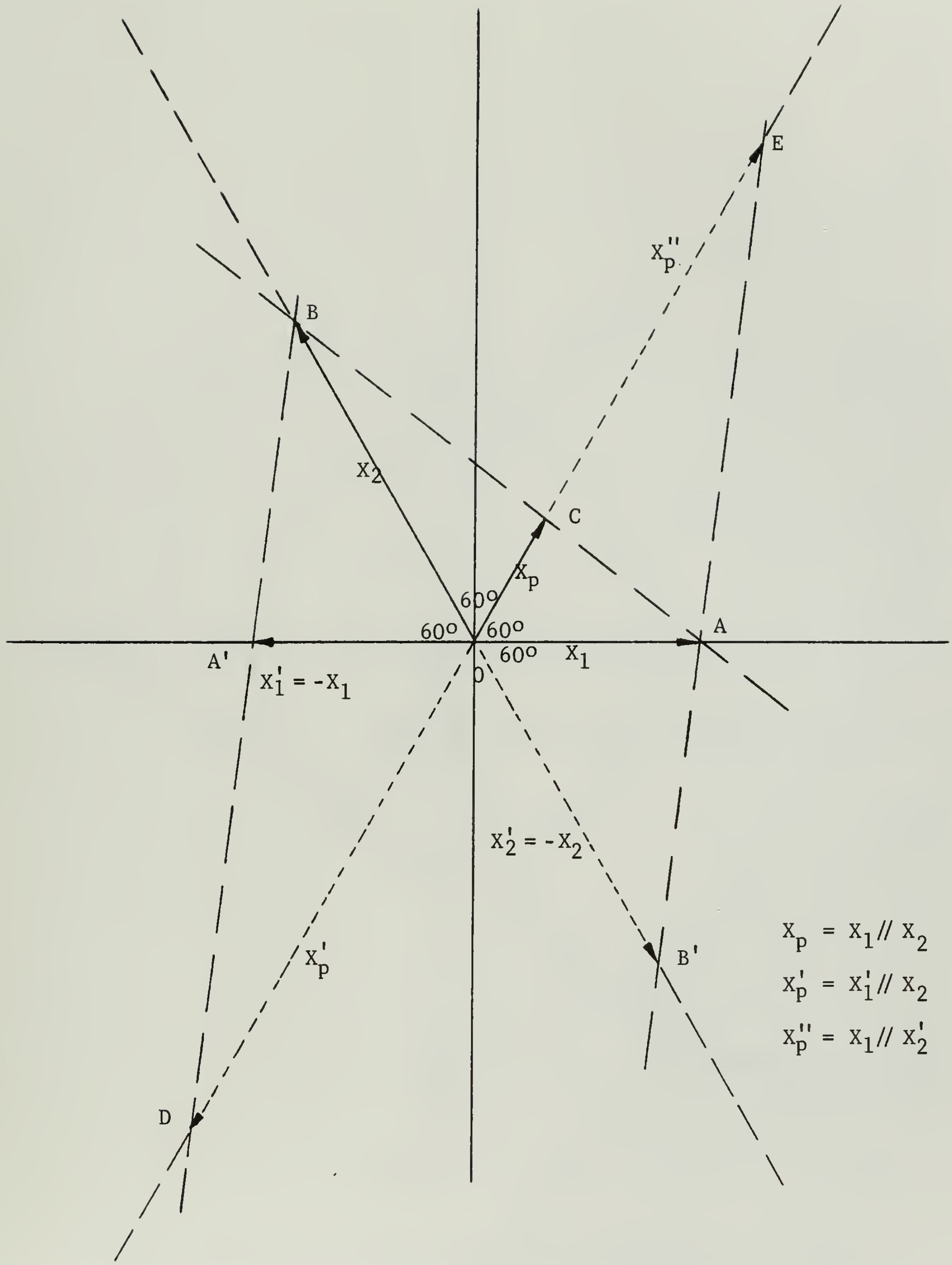


Figure 3-3. Impedances 180^o Apart, Axes Separated by 120^o

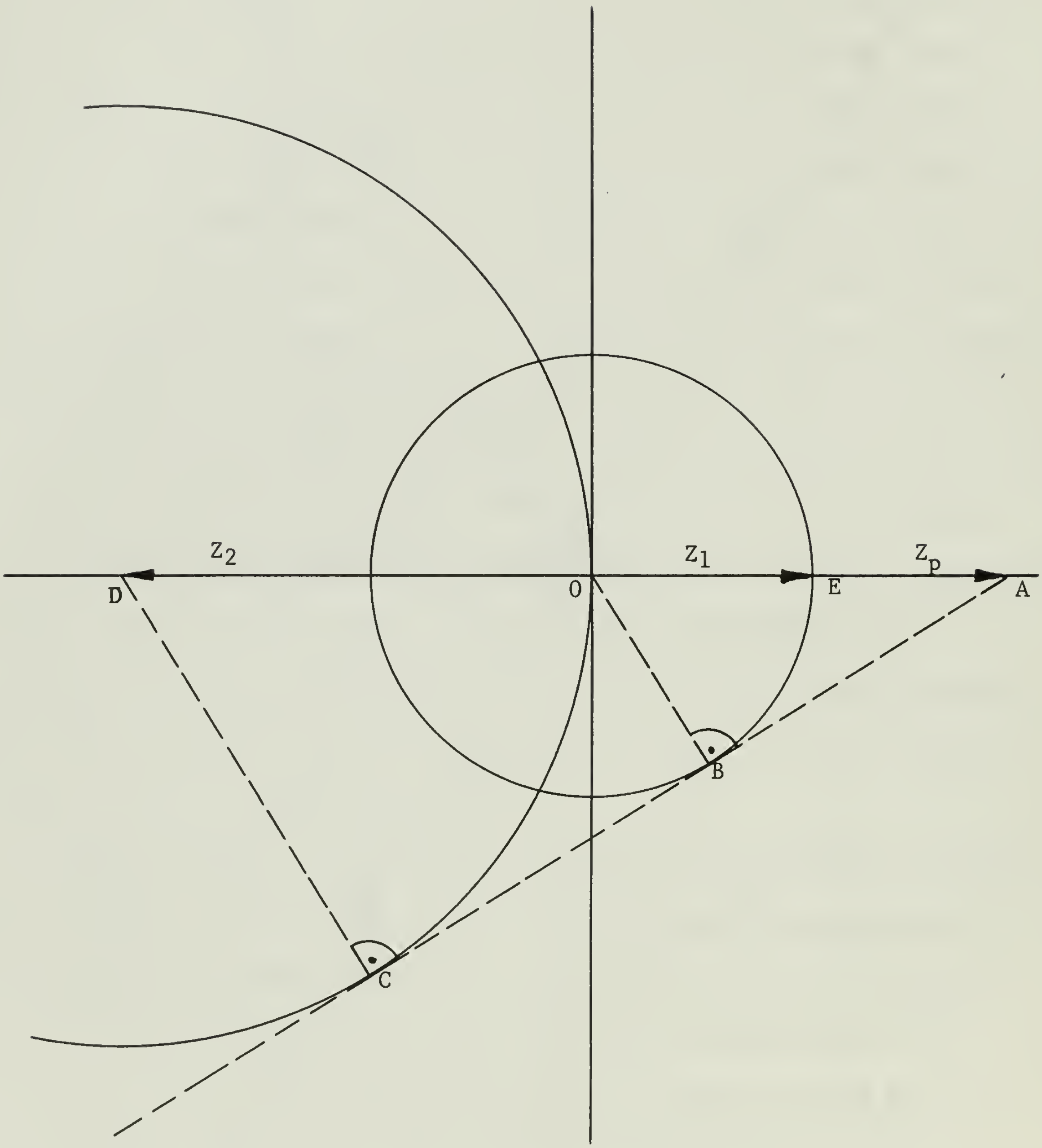


Figure 3-4. Impedances 180° Apart, Method Developed by Kind

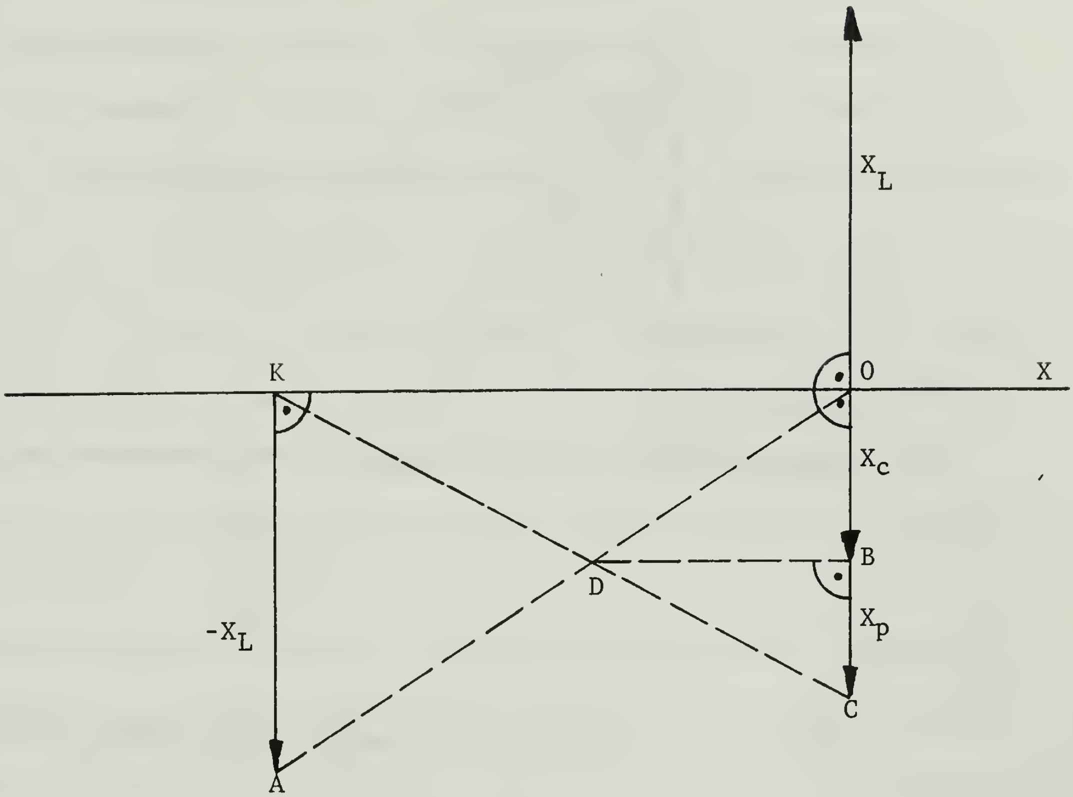


Figure 3-5. Impedances 180° Apart, Common Base Line of Arbitrary Length

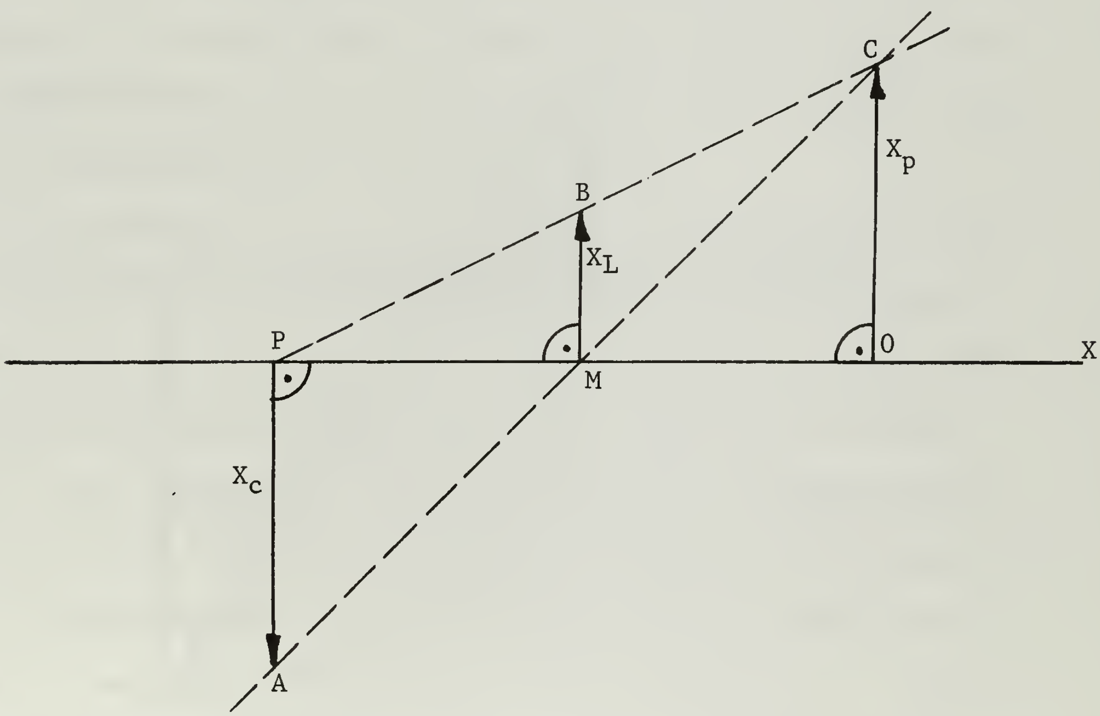


Figure 3-8. Impedances 180° Apart, Common Base Line of Arbitrary Length

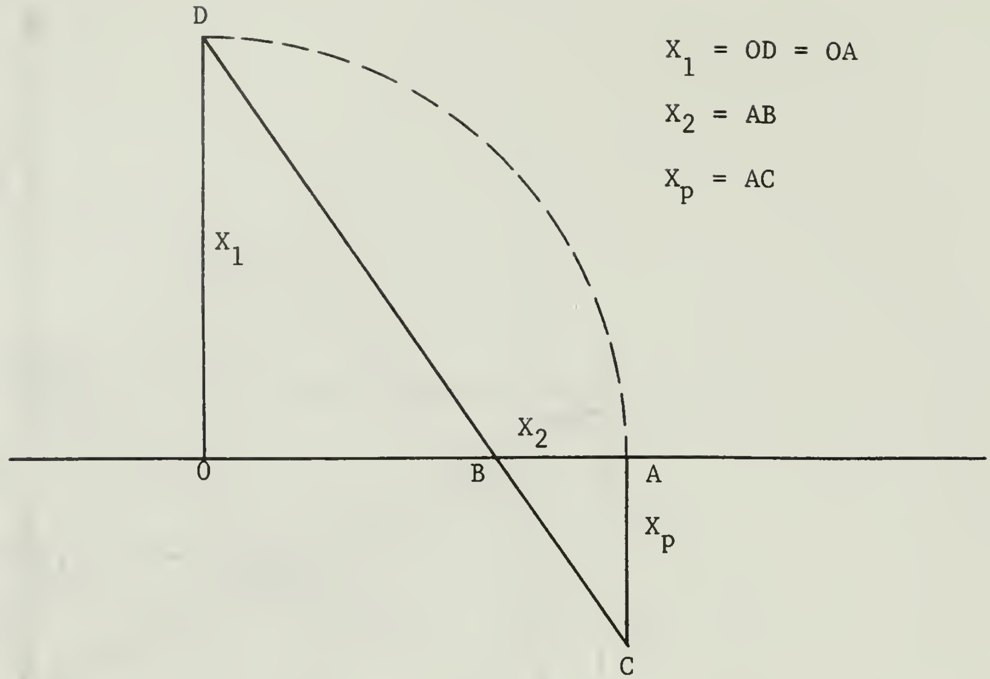


Figure 3-6. Impedances 180° Apart, Fixed Common Base Line

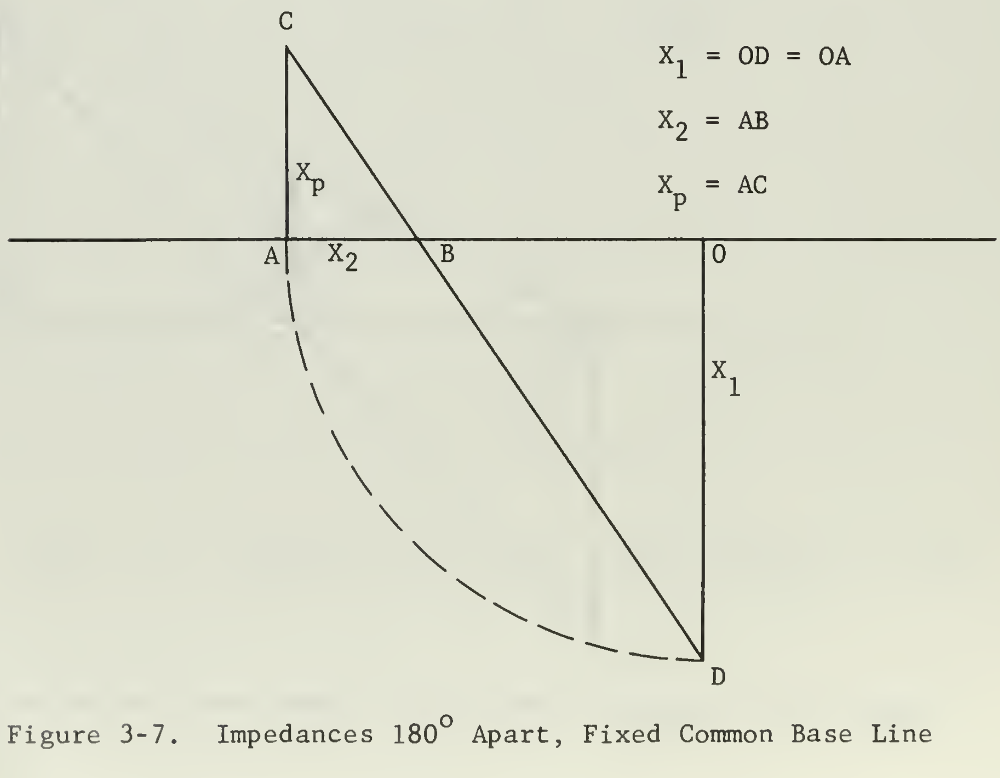


Figure 3-7. Impedances 180° Apart, Fixed Common Base Line

Figure 3-6 OA is an inductance, AB a capacitance, and in Figure 3-7 OA is the capacitance and AB the inductance. Since in both cases the phasor OA represents X_1 and the triangles CAB and DOA are similar, the relations AC/AB = OD/OB from Figure 3-6 lead to the resultant equivalent impedance $X_p = + \frac{X_1 X_2}{X_1 + X_2}$, given by the phasor AC in both cases.

Figure 3-8 depicts a very similar method, published by D. W. Wells. X_l and X_c are drawn perpendicular to a common base line, separated by any distance PM and with their proper respective phase angle. The intersection of the extended lines AM and PB gives point C. OC represents the parallel equivalent X_p.

Rulop introduces a further method for paralleling two impedances 180 degrees in phase apart that is very similar to his method for like impedances (Figure 2-1). In Figure 3-9 X_l and X_c are drawn perpendicular to the x-axis. Two sets of parallel intersecting lines are sketched through the tips and the common end point of X_l and X_c and line MP is laid through the intersections of lines a with b' and a' with b. OC is then the parallel value X_p in magnitude and phase.

A special case to the more general method described above is outlined in Figure 3-10. One set of parallel lines, b and b', is drawn perpendicularly, the other set, a and a', at a 45 degree angle to the given reactances. X_p is then obtained in the same manner as before.

H. P. Hall introduced a method to parallel two impedances 180 degrees out of phase that does not require any transfer or measuring of a phasor. As shown in Figure 3-11, a 45 degree line a is drawn in the quadrant of the smaller impedance, here X_c, originating in 0. Line "a" serves to get point B', where OB = $|X_c|$. Then the extended line AB' crosses line a at point D and the perpendicular to the y-axis through D

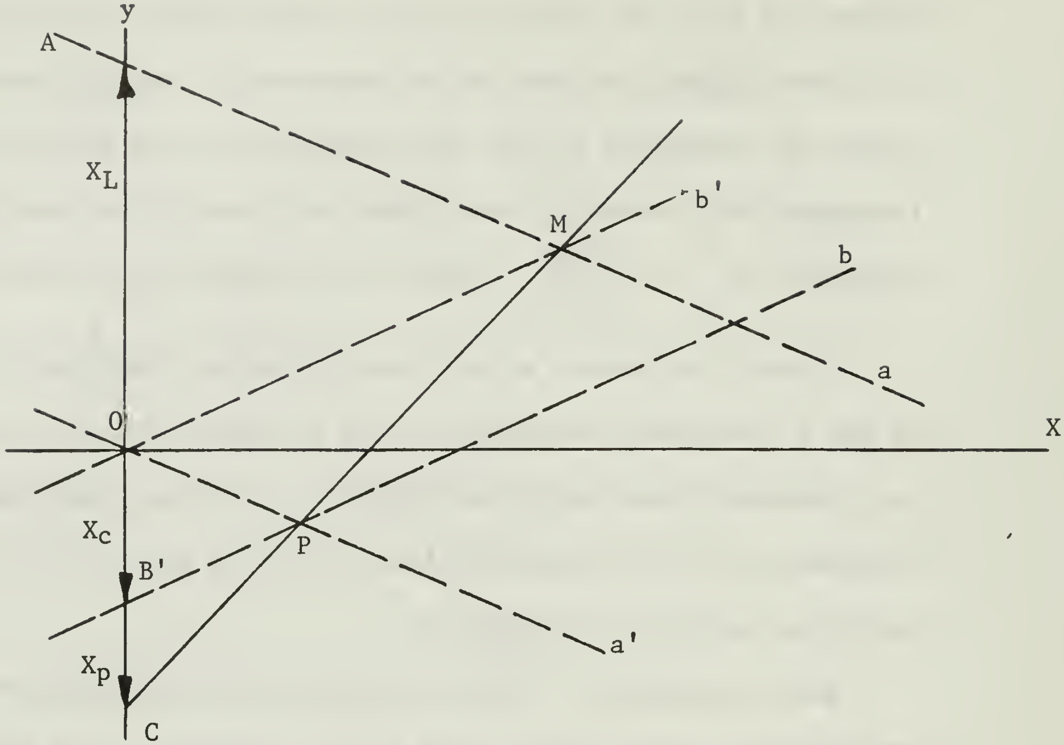


Figure 3-9. Impedances 180° Apart, Method by Rukop, Using Parallel Sets of Lines

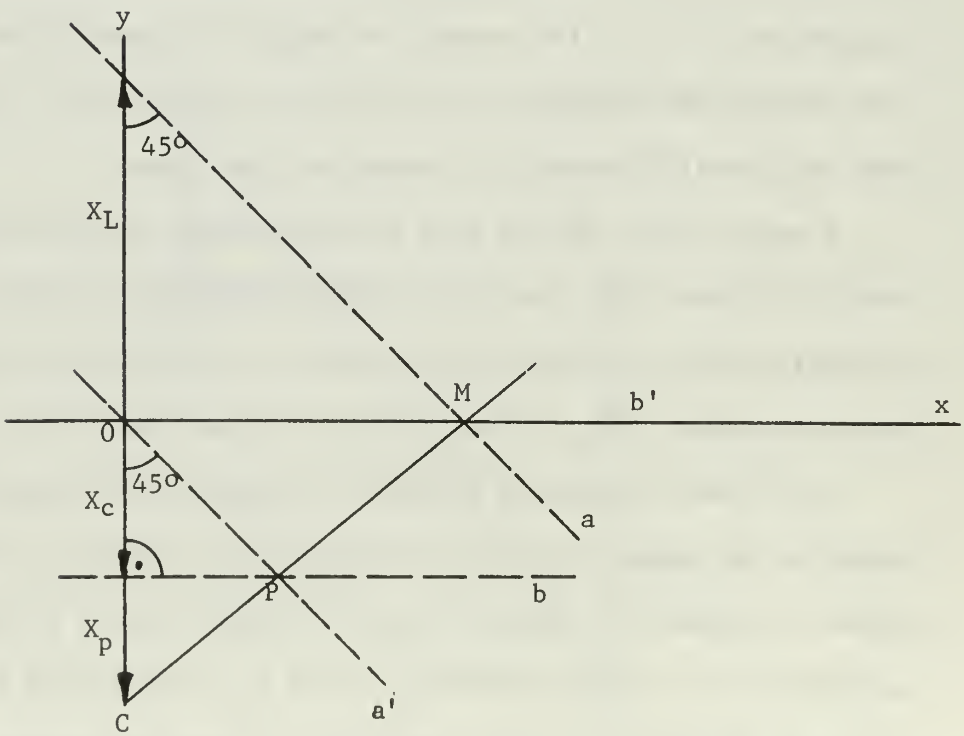


Figure 3-10. Impedances 180° Apart, Method by Rukop, Simplified Construction

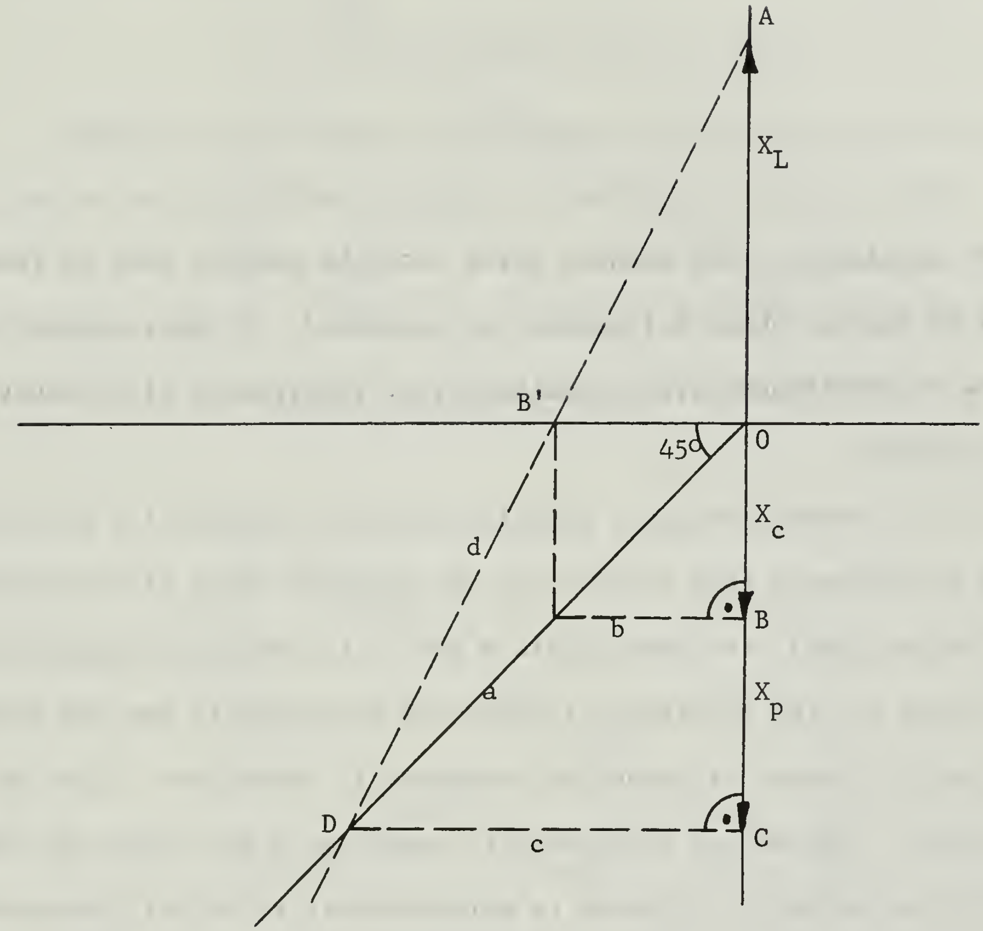


Figure 3-11. Impedances 180° Apart, Construction by Hall

gives C. OC is the resultant impedance X_p in magnitude and phase. The proof of this method is presented in Appendix 7.

4. The Paralleling of Two Impedances 90 Degrees Out of Phase.

Before entering the general field of paralleling two or more arbitrary impedances, five methods going into the special case of impedances with 90 degrees phase difference are presented. In this chapter combinations of resistances with capacitances or resistances with inductances are treated.

E. C. Goodale brings a graphical method to parallel a pure reactance with a resistance that yields only the absolute value of the parallel equivalent, while the phase angle θ has to be calculated separately. In Figure 4-1 the resistance is laid off horizontally and the reactance vertically, upward if inductive, downward if capacitive. Line AB is sketched. Then an arc is drawn with center at B and radius OB that cuts line AB at point C. CD which is perpendicular to OB will represent $|Z_p|$, the magnitude of the equivalent impedance. The resistive and reactive components of Z_p can be found easily and are given by CE and DE respectively. From the real and imaginary parts of Z_p the phase angle θ can calculated according to $\theta = \tan^{-1} \dfrac{\mathrm{Im}(Z_p)}{\mathrm{Re}(Z_p)}$ and because the two angles OAB and DCB are the same, θ can be found from the given values R and X according to $\theta = \tan^{-1} \dfrac{R}{X}$.

The next method to get $|Z_p|$ is described in Figure 4-2. R^2 and X^2 are laid off perpendicularly to the base line OE, where OA and EB can be separated by any distance OE. The intersection of the cross diagonals gives point C and DC, perpendicular to OE, yields $|Z_p|^2$. The phase angle θ of Z_p does not appear at all in this construction but can be

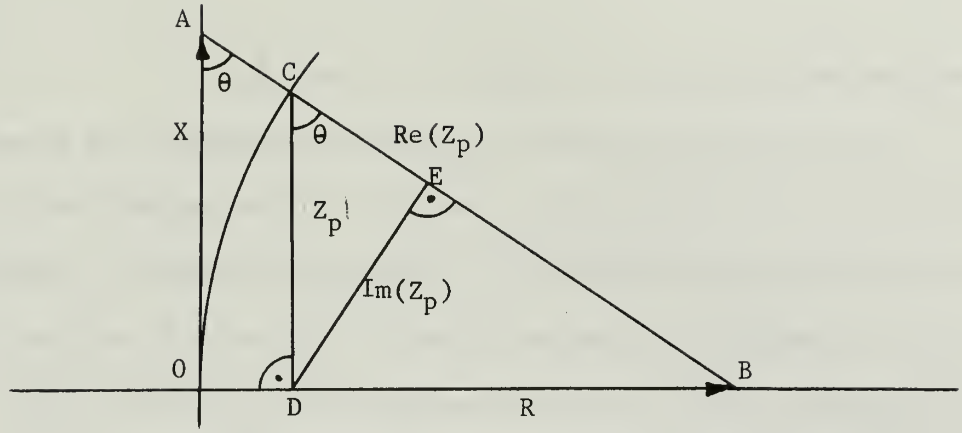


Figure 4-1. Impedances 90° Apart, Method by Goodale, Yielding Magnitude Only

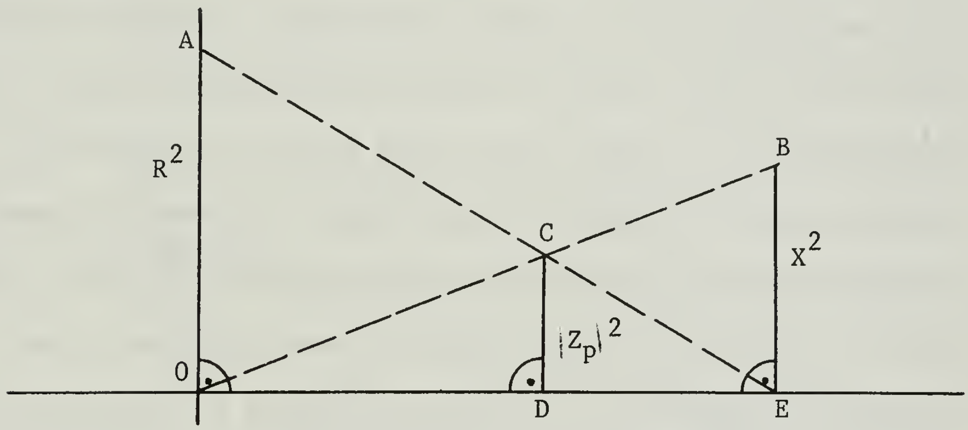


Figure 4-2. Impedances 90° Apart, Common Arbitrary Base Line and Yielding Magnitude Only

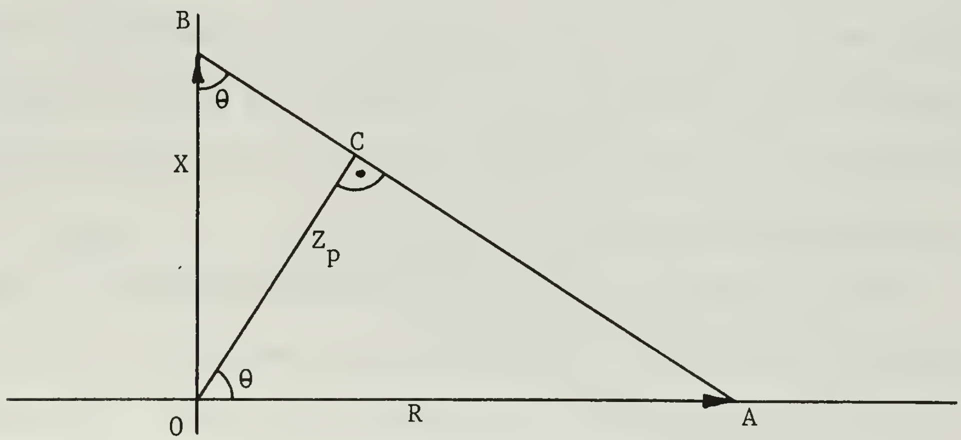


Figure 4-3. Impedances 90° Apart, Applying Method Depicted in Figure 1-1

computed easily from the relation $\theta = \tan^{-1} \frac{R}{X}$.

In the following method R is drawn horizontally and X vertically, as can be seen in Figure 4-3. Through 0 a line perpendicular to AB is laid forming the intersection C. OC then represents Z_p, the parallel value, in magnitude and proper phase θ. Figure 4-3 reaffirms the previously found relation for the phase angle of Z_p, $\theta = \tan^{-1} \frac{R}{X}$. The proofs for the above three methods are outlined in Appendix 8.

A fourth case falling into the category of resistance in parallel with a pure reactance is treated in Boening's paper and depicted in Figure 4-4. Given R and X, a circle with arbitrary center, but passing through D and H, the tips of the R and X vector respectively, is drawn. Since the exterior segments of the secants are inversely proportional to their corresponding secants, the relation holds $(OD)\cdot(OA) = (OG)\cdot(OH) = (OT)^2 = 1$. Now OG and OD are added vectorally to give OF and a circle is sketched, centered at 0 and passing through F, cutting the original circle at B, and thus forming a third secant. OA is measured along OF to yield OE, which is clearly the parallel impedance Z_p. To check the final result, D and H can be connected and a perpendicular from 0 can be dropped to DH yielding $OE = Z_p$. The method of Boening illustrates the application of tangent and secant theorems. The proof for this method is given in Appendix 15.

The last method in this chapter originates with R. C. Paine, who published it in 1942. He applies the following principle of geometry to parallel two impedances 90 degrees in phase apart. The principle states: The perpendicular from the vertex of the right angle to the hypotenuse of a right angle triangle is the mean proportional between the thereby created two segments of the hypotenuse. In Figure 4-5 a

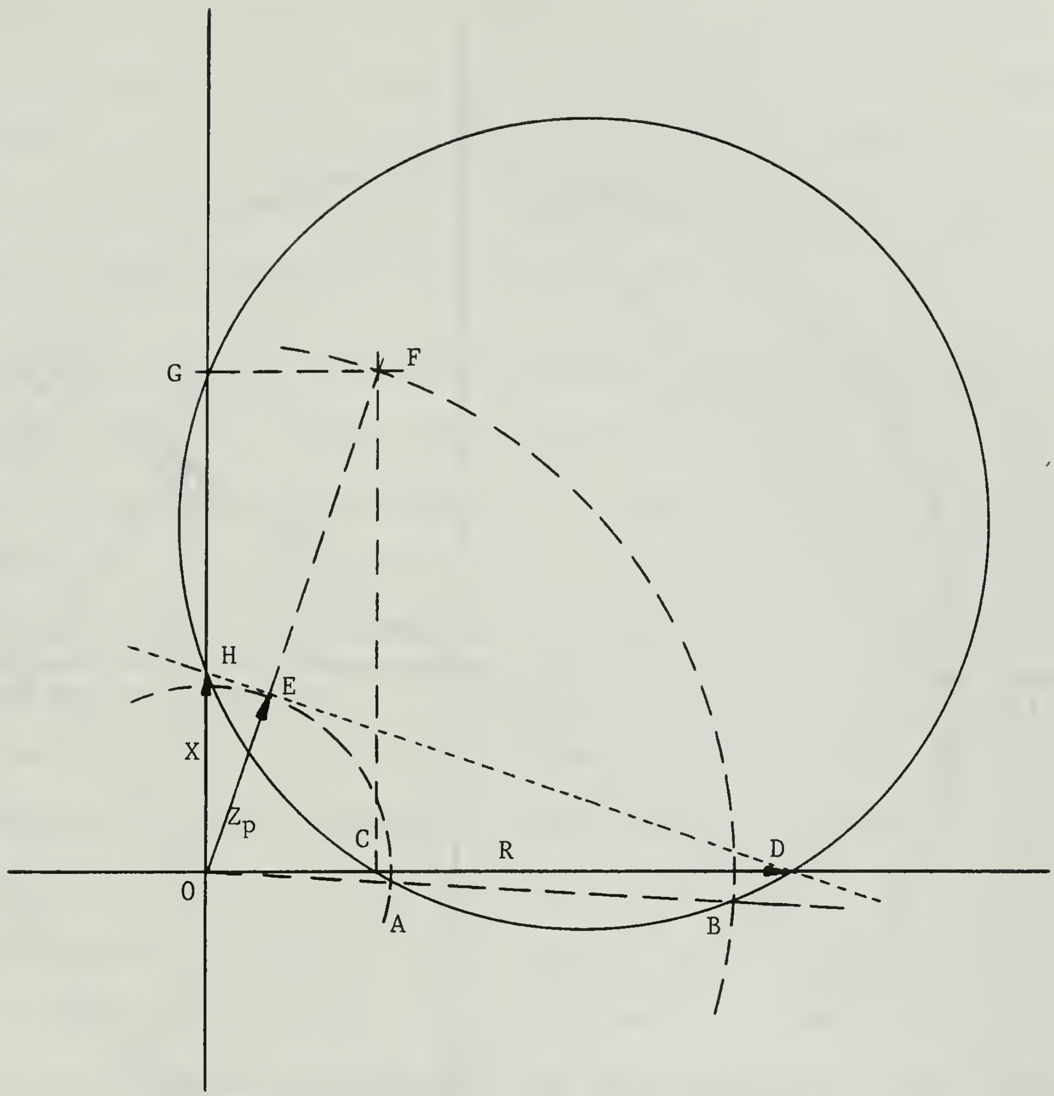


Figure 4-4. Impedances 90° Apart, Employing Secant and Tangent Theorems

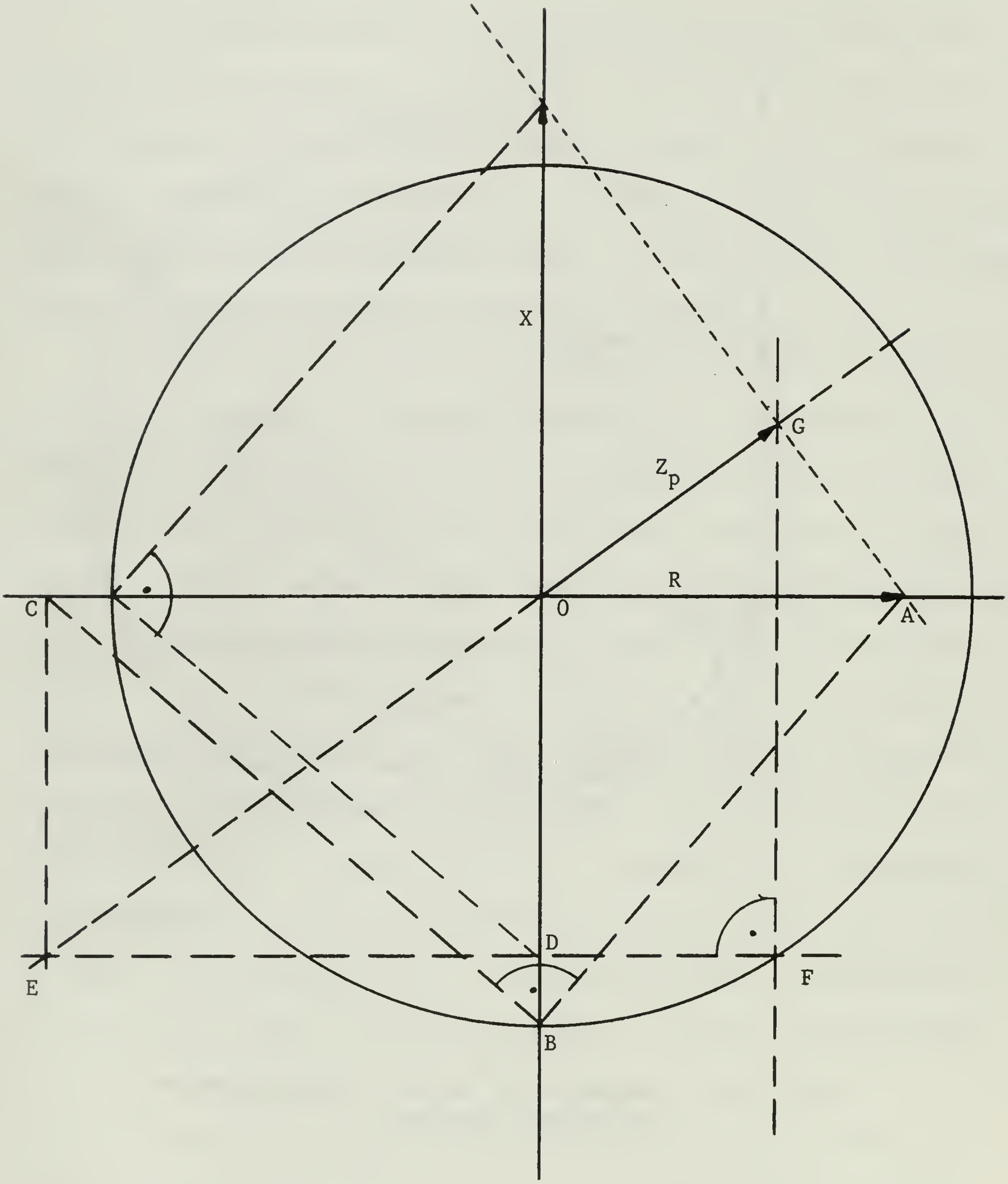


Figure 4-5. Impedances 90° Apart, Construction by Means of the Mean Proportionality Theorem

circle is drawn centered at O with any convenient radius. At point B a right angle ABC is formed, A representing the tip of R. If OB is normalized and set equal to one, OC will be 1/R. The same is done with X to get OD equal to 1/X. Then OD and OC are added vectorally to yield OE, which has to be inverted to OG with the help of the right triangle EFG. OG represents the equivalent impedance Z_p. The proof of this method is given in connection with Figure 5-12.

5. The Paralleling of Two Impedances Separated by Any Phase Angle.

This chapter is the most voluminous one, because the general case of two or more impedances separated by any phase angle is the most interesting one to electrical engineers. The previous chapters all dealt with special cases of this category, its methods therefore were mostly simpler and less involved.

The first two methods in this chapter are concerned with finding the inverse of a complex number in general, a problem encountered very frequently in the area of electrical engineering. The next group of methods explores constructions with straight lines only, followed by a group of methods where circles are used to parallel impedances. Thereafter some special methods for very acute or obtuse angles between the impedances and four cases where one impedance is a pure resistance are developed.

The reciprocal of a complex number can be found in a rather easy fashion with a method pointed out by Sylvan and depicted in Figure 5-1. Given is a complex number represented by the vector Z. A circle centered at O is drawn to intersect Z at point C. Then a second circle centered anywhere on the real axis to the right of B and going through C is constructed forming point F where it cuts the original circle.

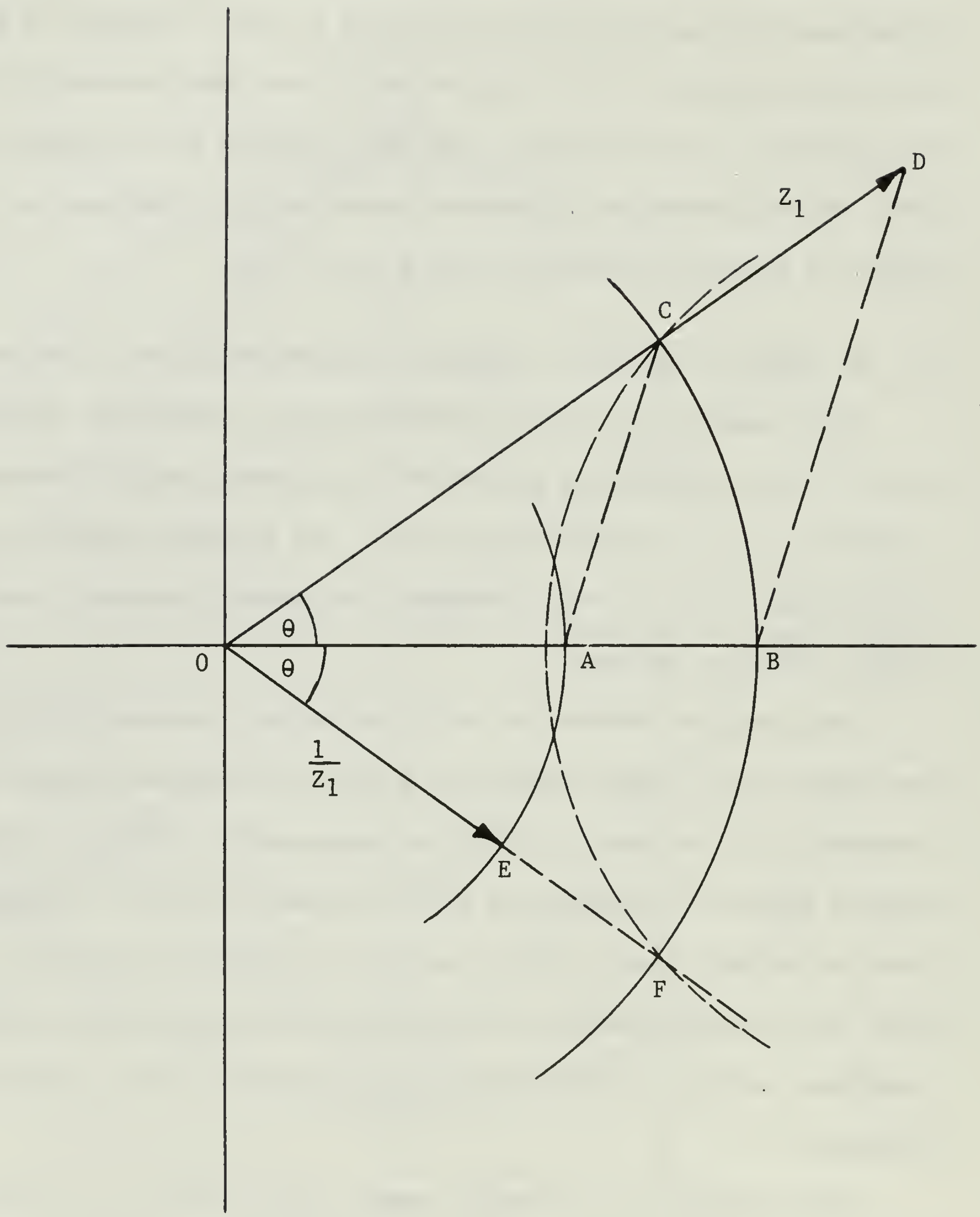


Figure 5-1. Inversion of a Complex Number, Method by Sylvan

Line OF is the reflection of line OD about the x-axis. Line DB is drawn and a parallel to it passing through C, thus yielding point A on the real axis. A circle of radius OA and centered at O will lead to point E on OF. OE then represents 1/Z, the wanted inverse of Z. The proof is offered in Appendix 9. The above method can be extended to add reciprocals of several complex numbers or impedances and to invert the final result to get the impedance of several impedances in parallel.

The second method to invert a complex number Z is presented by Boehne in connection with Figure 5-2. For a given Z, assumed to be an impedance, first its parallel components X_p and R_p are determined. Then points (-1 , 0) and (0 , -1) are located on the real and imaginary axis respectively to the scale of Z. To fix point C" a perpendicular to line CD is erected at D, where D has the coordinates (0 , -1), and the intersection with the real axis has to be found. OC" represents the negative inverse of R_p namely -G and then OC' is to be G. Similarly point B' is gotten on the imaginary axis with OF' representing jB. Adding G and B vectorally results in the desired Y = 1/Z.

One of the first methods to parallel two impedances, but surely one of the most elegant ones originates with Rukop and is presented in Figure 5-3. For this construction first two pairs of parallel lines, normal to the given phasors Z_1 and Z_2, are needed. OP and GB are normal to Z_1 and pass through its tip and end respectively, while OM and BC are normal to Z_2 and pass through its tip and end. The perpendicular OD to line MP represents the desired parallel equivalent Z_p. The proof is elaborated in Appendix 10. This method was republished later by Landolt, Paine, Batchelder and Bruehne. It must be noted that this method is good if Z_1 and Z_2 bear a large angle with respect to each other. For

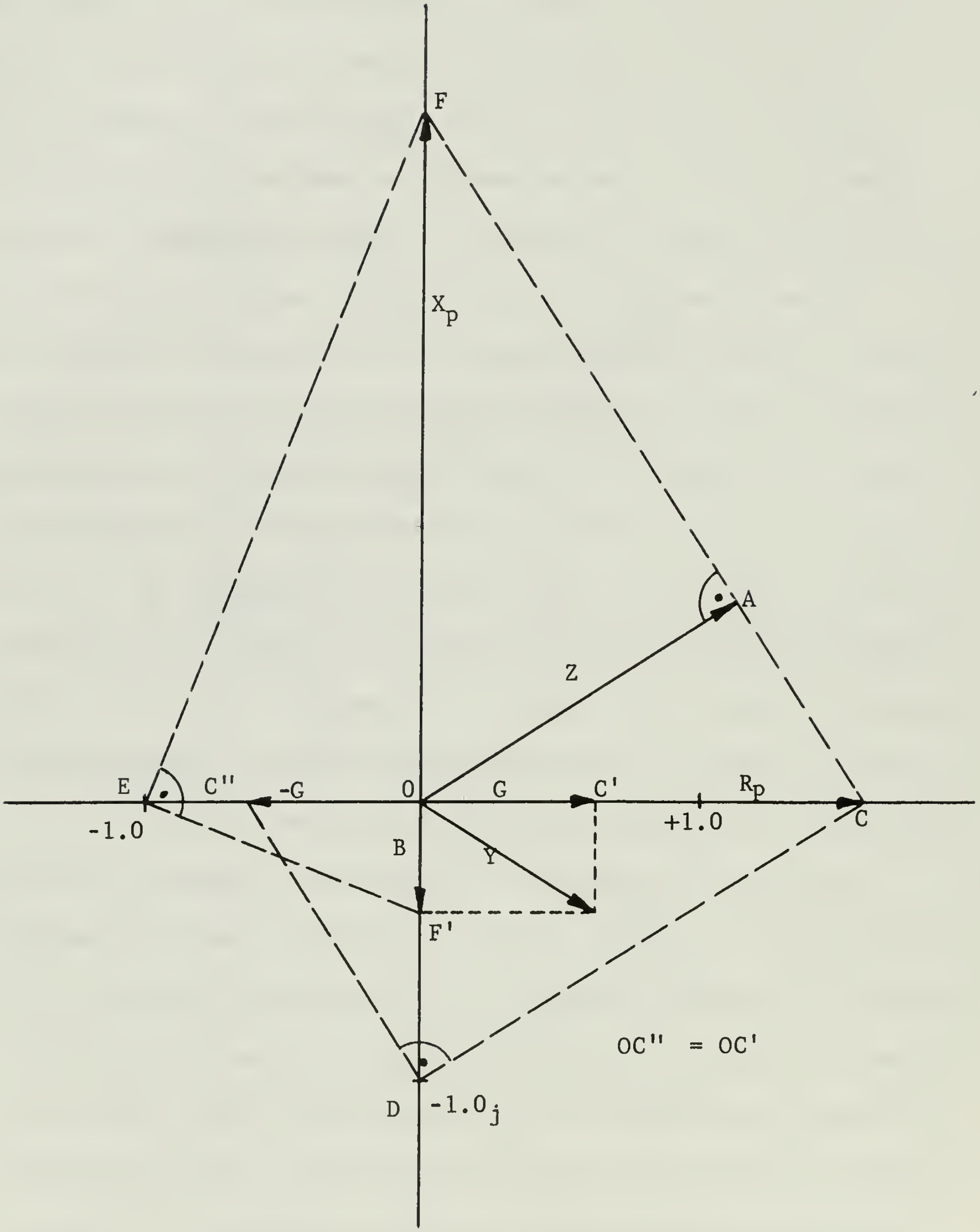


Figure 5-2. Inversion of a Complex Number, Method by Boehne

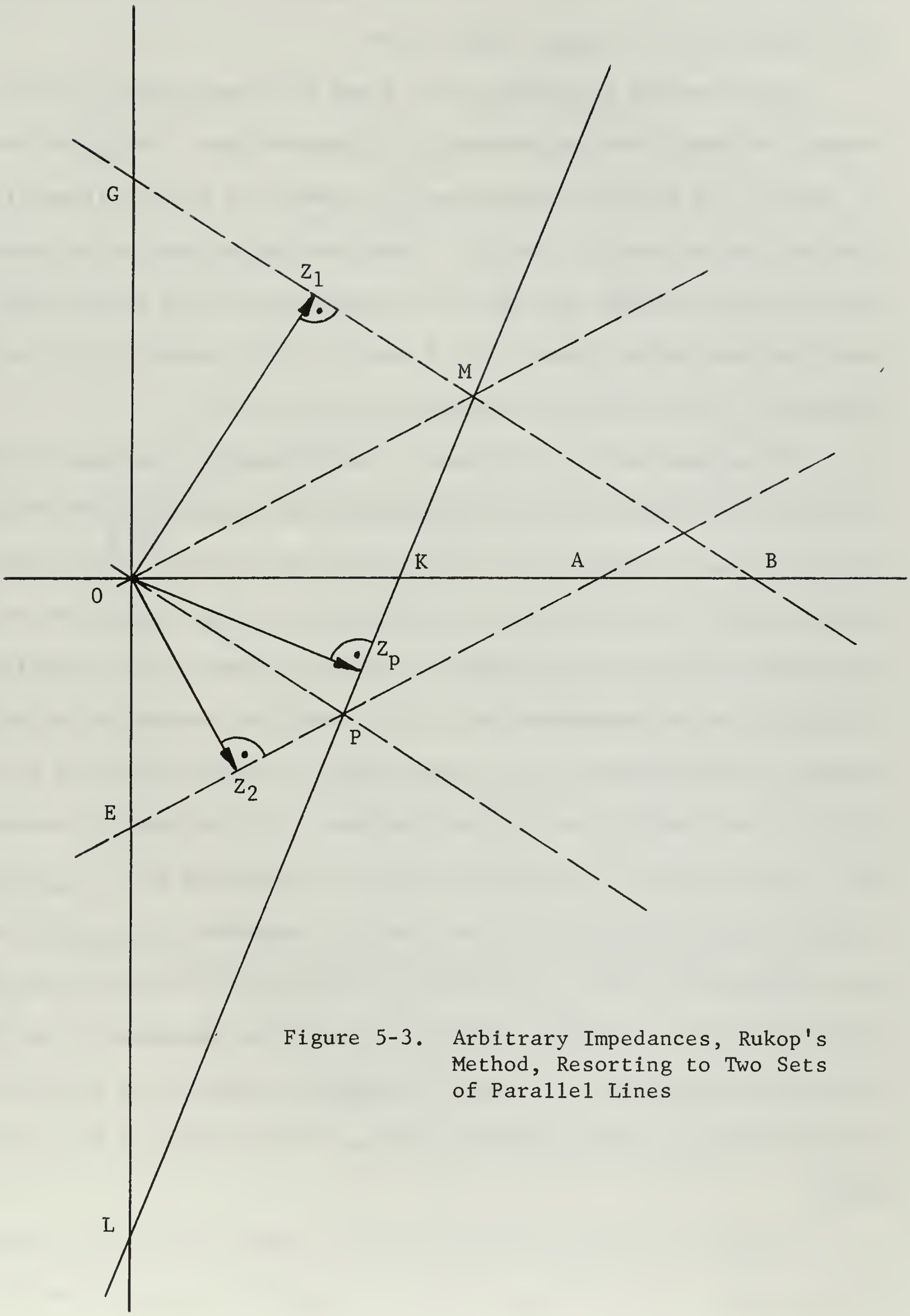


Figure 5-3. Arbitrary Impedances, Rukop's Method, Resorting to Two Sets of Parallel Lines

phasors which are close together this method consumes too much space and the method given in Figure 5-18 is more appropriate because it brings the construction in a more compact form.

Another method originating with Rukop but republished later by Boehne, Horwood, Reed and Edwards is introduced next. In Figure 5-4 Z_1 and Z_2, the two given impedances, are added and the resulting triangle ODB has the two angles α and β. These two angles have to be measured and a similar triangle OAC has to be constructed on one of the sides with the same angles, where $\alpha' = \alpha$ and $\beta' = \beta$. Phasor OC is then the desired Z_p. The proof is developed in Appendix 11.

In the construction of Figure 5-5 Boehne applies the same principle as he does in Figure 5-9 but he simplifies the construction by the proper choice of axes. Boehne uses the fact that the parallel value of two phasors bears a fixed relation to its parents, it is independent of the coordinate axis used to determine its magnitude and relative position. In Figure 5-5 the coordinate axis x' is chosen to coincide with the resistance intersection R, or in other words, the coordinate axis is chosen in such a way that Z_1 and Z_2 have the same resistive parallel components OR. The equivalent of these two resistive components is R_p, represented by OM. The equivalent of the two reactive components X_{p1} and X_{p2} is X_p, represented by ON. In both cases the method of Figure 2-2 is applied to find R_p and X_p. From R_p and X_p the equivalent impedance Z_p is found according to Figure 1-1. It must be emphasized that R_p is no pure resistance and X_p no pure reactance because of the choice of the reference axes.

In Figure 5-6 Boehne displays another method that reduces necessary constructions. The reference axis x' is chosen to coincide with the

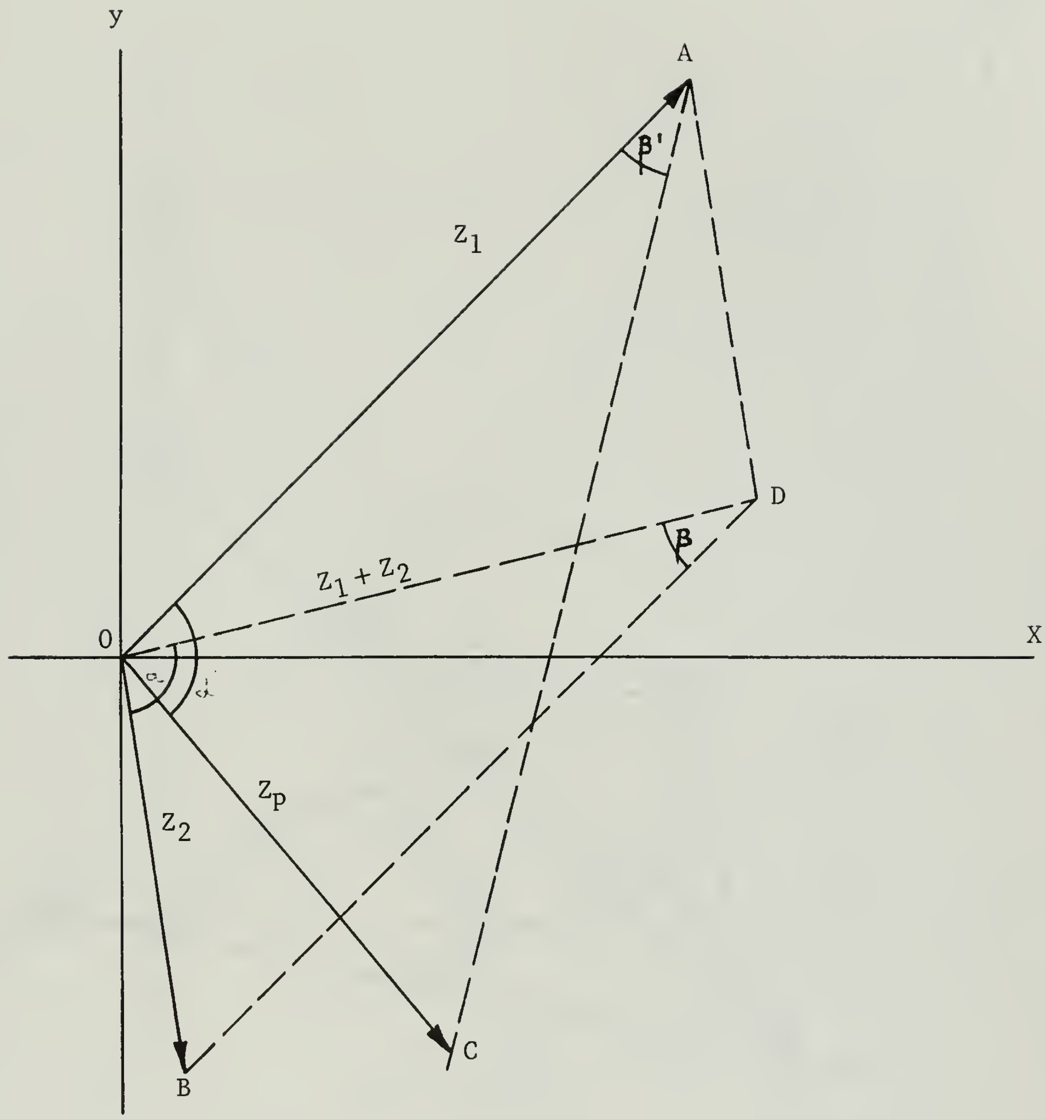
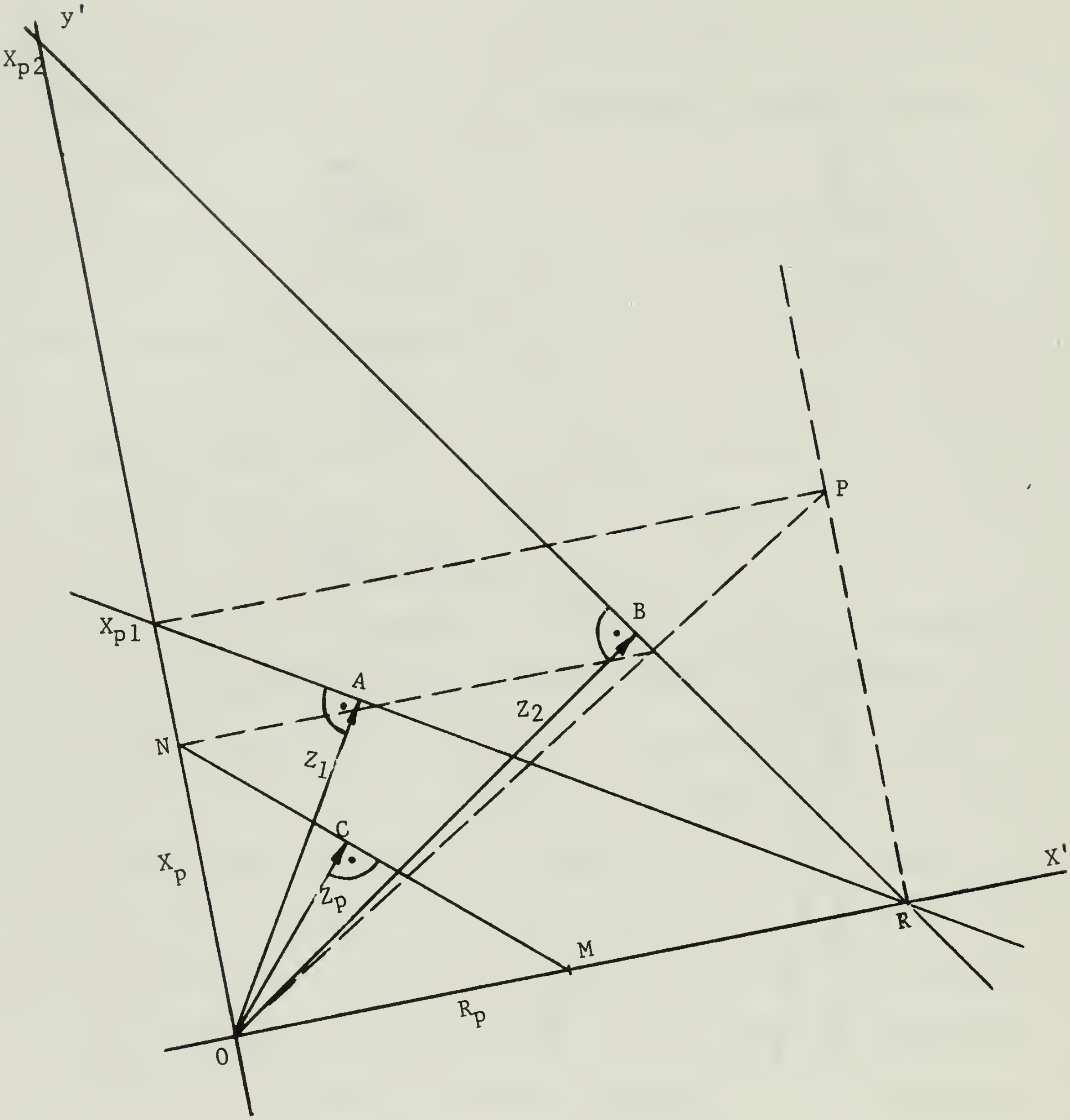


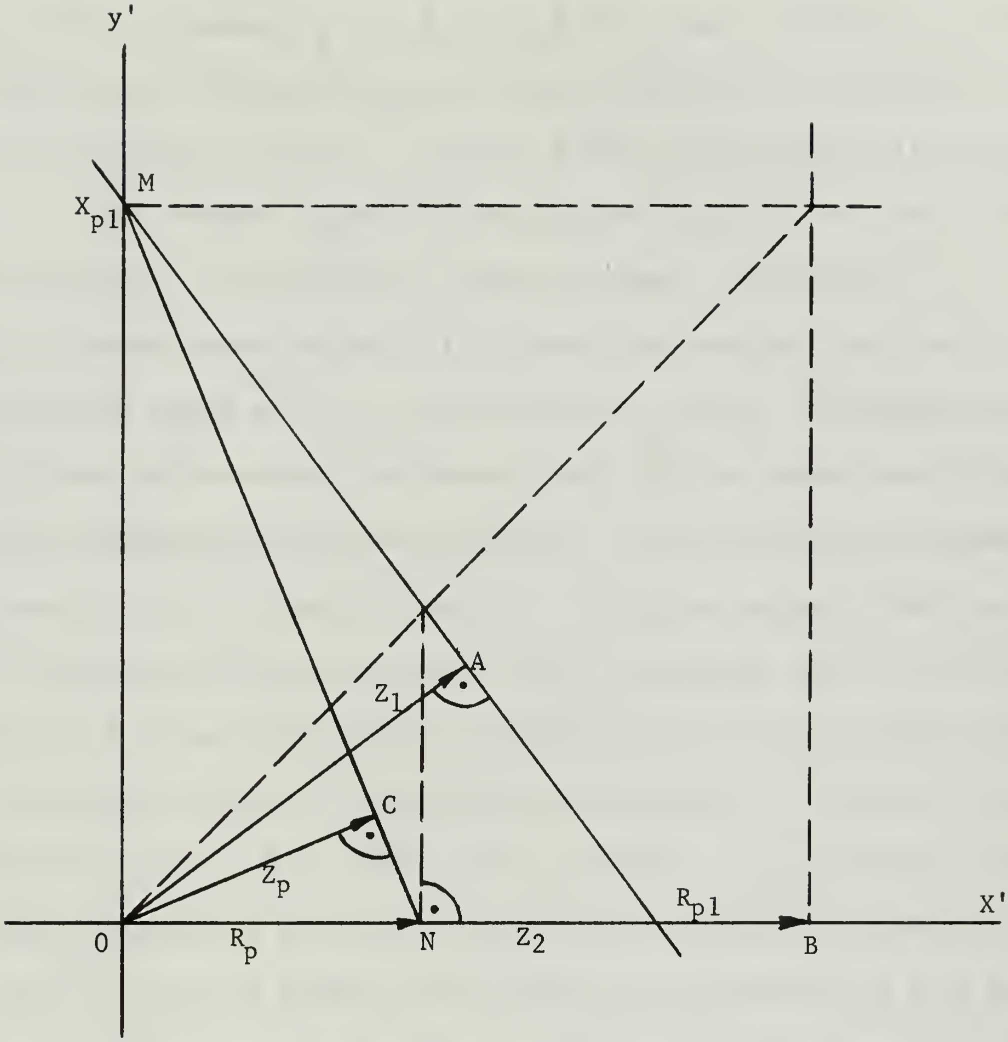
Figure 5-4. Arbitrary Impedances, Using Similar Triangles



OM = MR

OM = OR// OR

Figure 5-5. Arbitrary Impedances, Reference Axis is Chosen to Coincide with Resistance Intersection



$$R_p = Z_2 \,/\!/\, R_{p1}^{*}$$

Figure 5-6. Arbitrary Impedances, Reference Axis is Chosen to Coincide With Z_2

phasor Z_2, which now has only an R* component. Z_1 has to be transformed into its parallel components R_{p1} and X_{p1}. R_p represents then the parallel equivalent of Z_2 and R_{p1} and the desired parallel equivalent of Z_1 and Z_2 is formed easily from R_p and X_{p1}. In this construction, too, R_p and X_p are not pure resistances and reactances respectively.

In 1938 Goodale, another pioneer in the field of graphical methods in electrical engineering, came up with the following method to parallel two impedances. Figure 5-7 demonstrates it. The given impedances Z_1 and Z_2 are broken up into their respective resistive and reactive components R_1, R_2 and X_1, X_2. X_1 and X_2 are laid off along the negative y-axis and then perpendicular to OA and through A the resistances R_1 and R_2 are laid off forming AB. OB is chosen as new reference line. On the new reference axis R_2 is measured off from 0 to C and at C a perpendicular of length X_2 is erected to give point D. OD then corresponds to Z_2 at an angle $\theta_2 - \theta_s$, where θ_2 is the phase of Z_2 and θ_s is the phase of Z_1 and Z_2 in series. On the line OD next R_1 is measured from 0 to F and at F a perpendicular is erected with length X_1 to give point G. OG then corresponds to Z_1 at a phase angle $\theta_1 + \theta_2 - \theta_s$, which is the proper phase angle of Z_p. A circle through B and centered at 0 is drawn and intersects the extension of Od at E. Points E and G are connected and a parallel to EG going through D yields H. OH then represents Z_p in magnitude and phase. The proof of this method can be found in Appendix 12.

The next method is well known. It was first mentioned by Reed in 1951 and later republished again by Horwood and Boehne. It is described in Figure 5-8. From the given Z_1 and Z_2 the series impedance Z_s is found by vector addition and is represented by OC. A circle is sketched

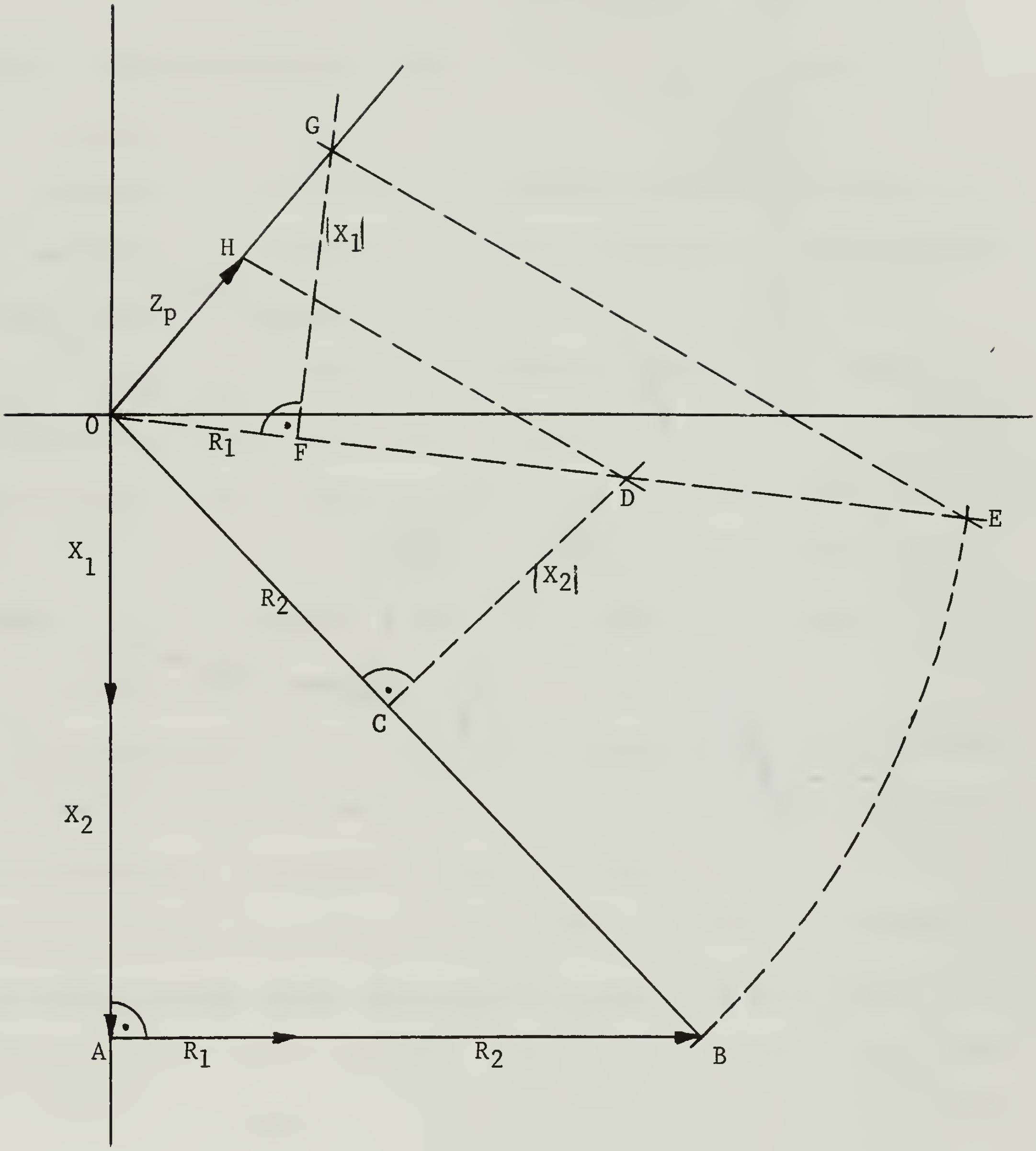


Figure 5-7. Arbitrary Impedances, Method by Goodale

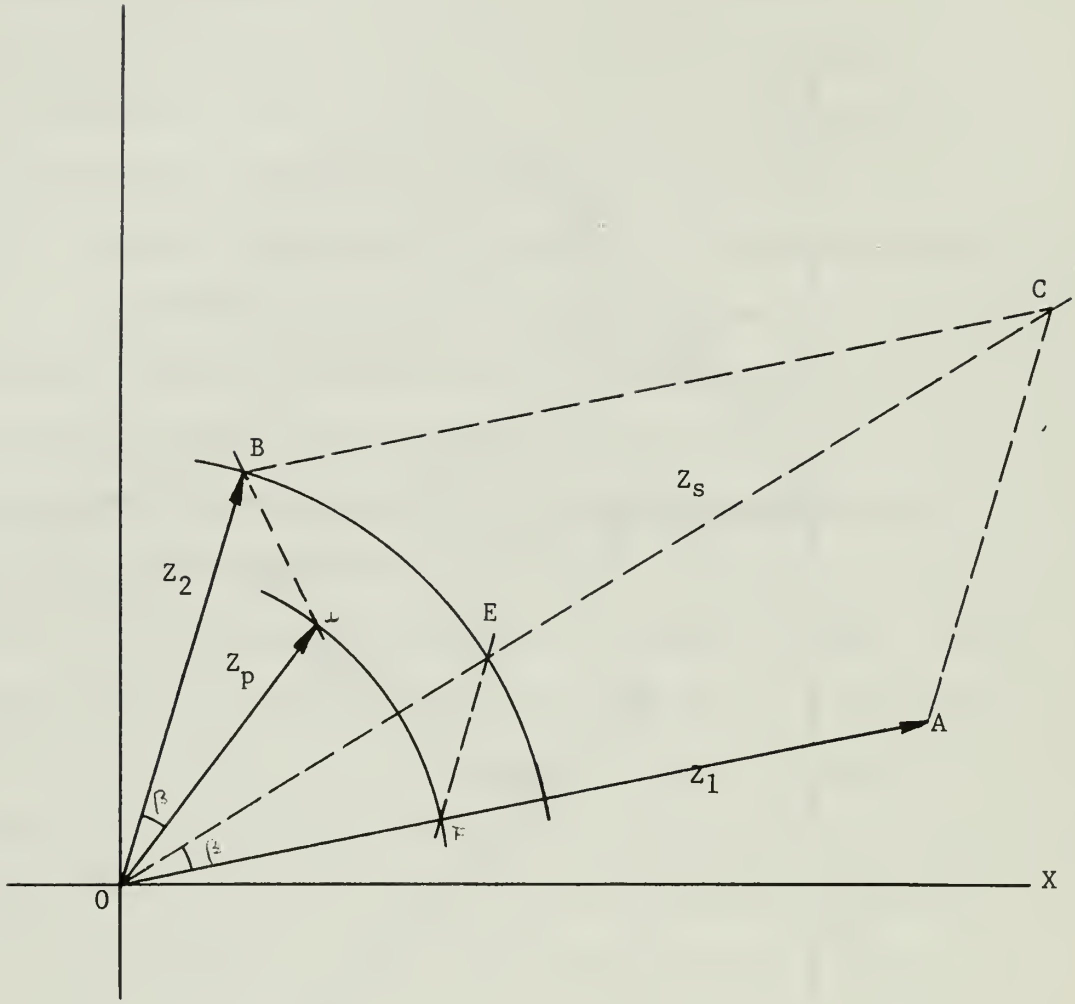


Figure 5-8. Arbitrary Impedances, Using Two Congruent Triangles

centered at 0 and passing through B and forming point E at its intersection with OC. The parallel to CA through E is drawn, intersecting OA in F. Angle β , formed by AOC, is measured off in the clockwise direction from line OB and the intersection with the circle drawn through F and centered at 0 yields D, with OD being the equivalent parallel value Z_p. The proof is demonstrated in Appendix 13.

Progressing to the section with methods using mostly circles for their constructions a method is presented first that has been published by L. Batchelder, J. Deignan, K. S. Kreielsheimer and Reppisch. As pictured in Figure 5-9 the given impedances Z_1 and Z_2 are indicated by the phasors OA and OB respectively. A circle tangent to OA at 0 and passing through B and a circle tangent to OB at 0 and passing through A have to be obtained. The common chord OC of these two circles represents the equivalent impedance value Z_p in magnitude and phase. The proof is worked out in Appendix 14.

The above method can be simplified with the aid of the principles used in Figure 5-23 and choosing a coordinate system that takes one of the two impedances as reference axis. In Figure 5-10 the reference axis is chosen to coincide with Z_2 , thus forming a rectangular coordinate system with the axes OB and OD. A circle tangent to OB at 0 and passing through A, the tip of Z_1, is drawn. OA is reflected about OD to form OA'. Line A'B intersects the circle in C and OC proves to be the desired Z_p.

Boening developed the next method which is based on the tangent, secant and the chord theorems. The tangent and secant theorems are stated in connection with Figure 2-11, while the chord theorem was connected with Figure 3-2. Referring to Figure 5-11, the construction is

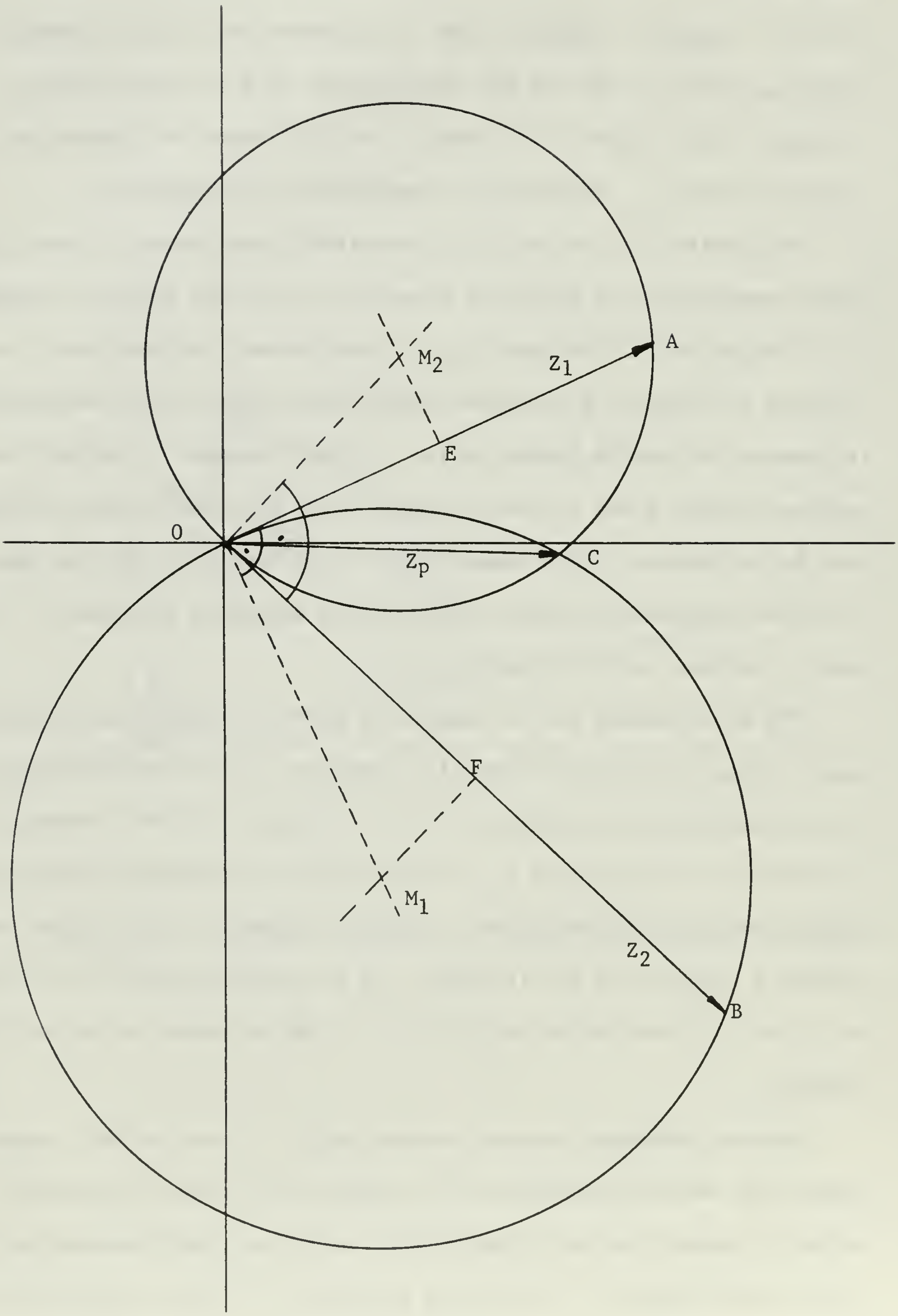


Figure 5-9. Arbitrary Impedances, Circle Method Using Common Chord

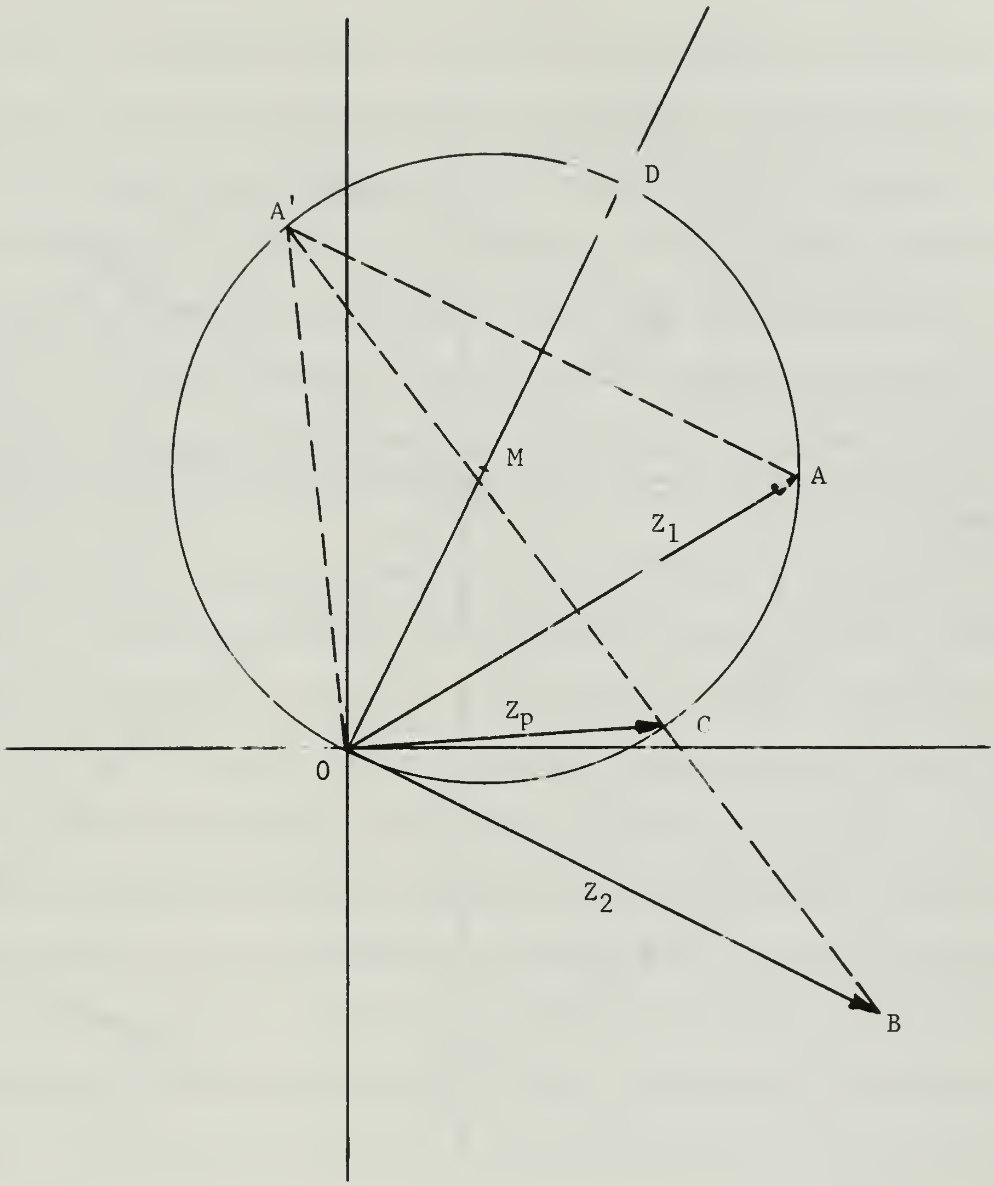


Figure 5-10. Arbitrary Impedances, Reference Axis Coinciding With Z_2

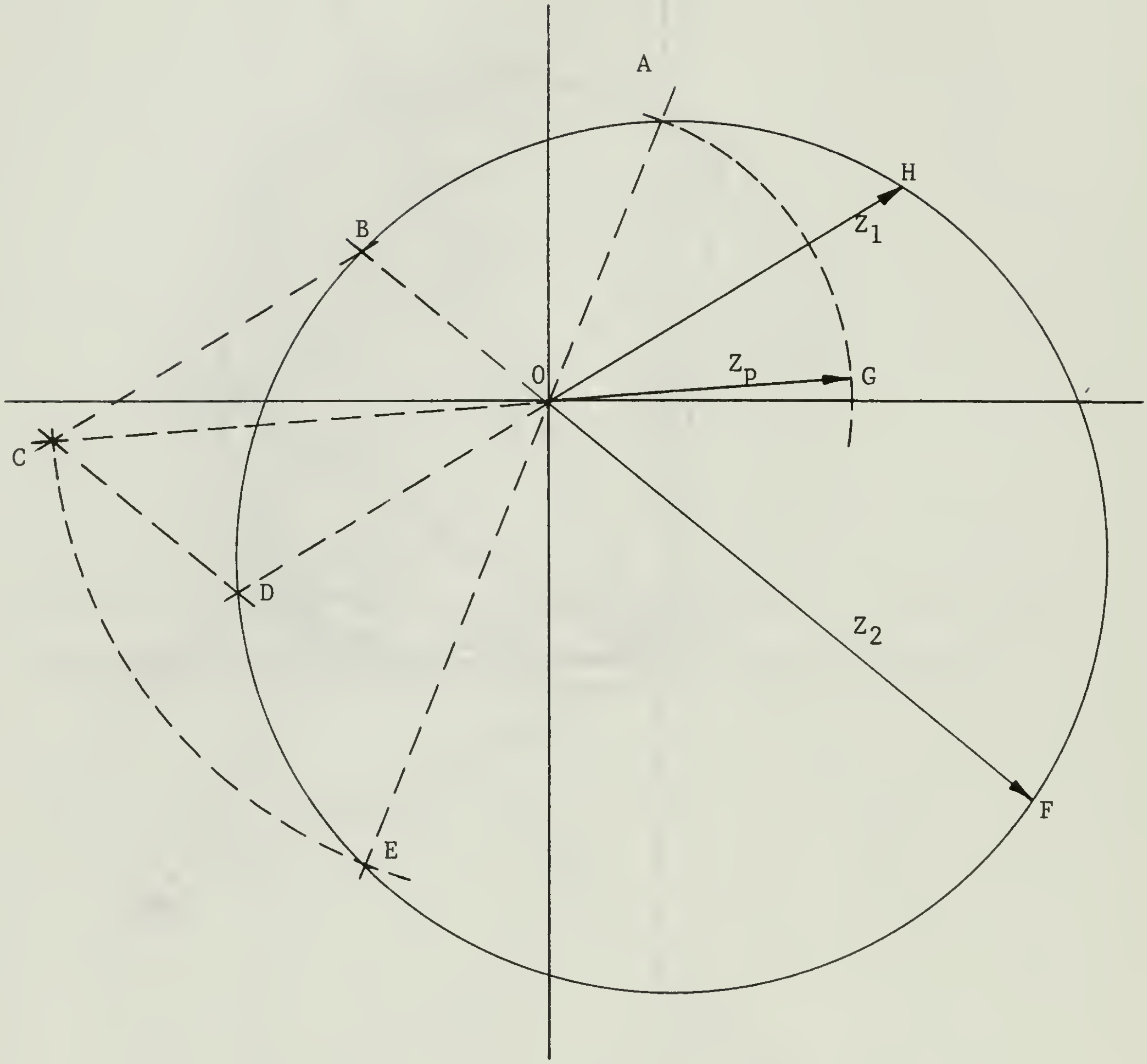


Figure 5-11. Arbitrary Impedances, Making Use of Secant, Tangent, and Chord Theorems

done as follows: A circle of arbitrary radius is drawn through the tips of the given impedance vectors Z_1 and Z_2. Then OH and OF are extended to form the lines DH and FB. OD and OB, the respective reciprocals of Z_1 and Z_2, must be added vectorally and yield OC. The intersection of the circle centered at O and passing through C with the original circle gives point E and line EO is extended to A. A third circle centered at O and passing through A is sketched and it cuts the extension of line OC at G. OG is the inversion of OC and thus the parallel equivalent Z_p. The proof of this method is shown in Appendix 15 in connection with Figure 5-12.

If the above method is applied to two impedances in the same quadrant, i.e. inductive or capacitive impedances only, then the solution becomes more obvious. In Figure 5-12 a circle is drawn through the tips of Z_1 and Z_2, so that both become secants, namely OD and OH respectively. By the secant theorem OG is the inverse of OH and OA the inverse of OD. Therefore OG and OA are added vectorally to give OF which is the inverse of Z_p. Now a circle is sketched, centered at O and passing through F, thus yielding point C. Line OC intersects the original circle at B. Then another circle centered at O and this time passing through B is drawn. It intersects line OF at E with OE being the desired equivalent impedance Z_p.

In 1953 Storch introduced a method using special graph paper, a paper on which a family of concentric circles about the origin is superimposed on a rectangular grid. Without this graph paper at hand the method is pretty complicated and does not show any specific advantage. Figure 5-13 gives a picture of it. In a rectangular coordinate system two circles with centers at the origin and passing through the tips of

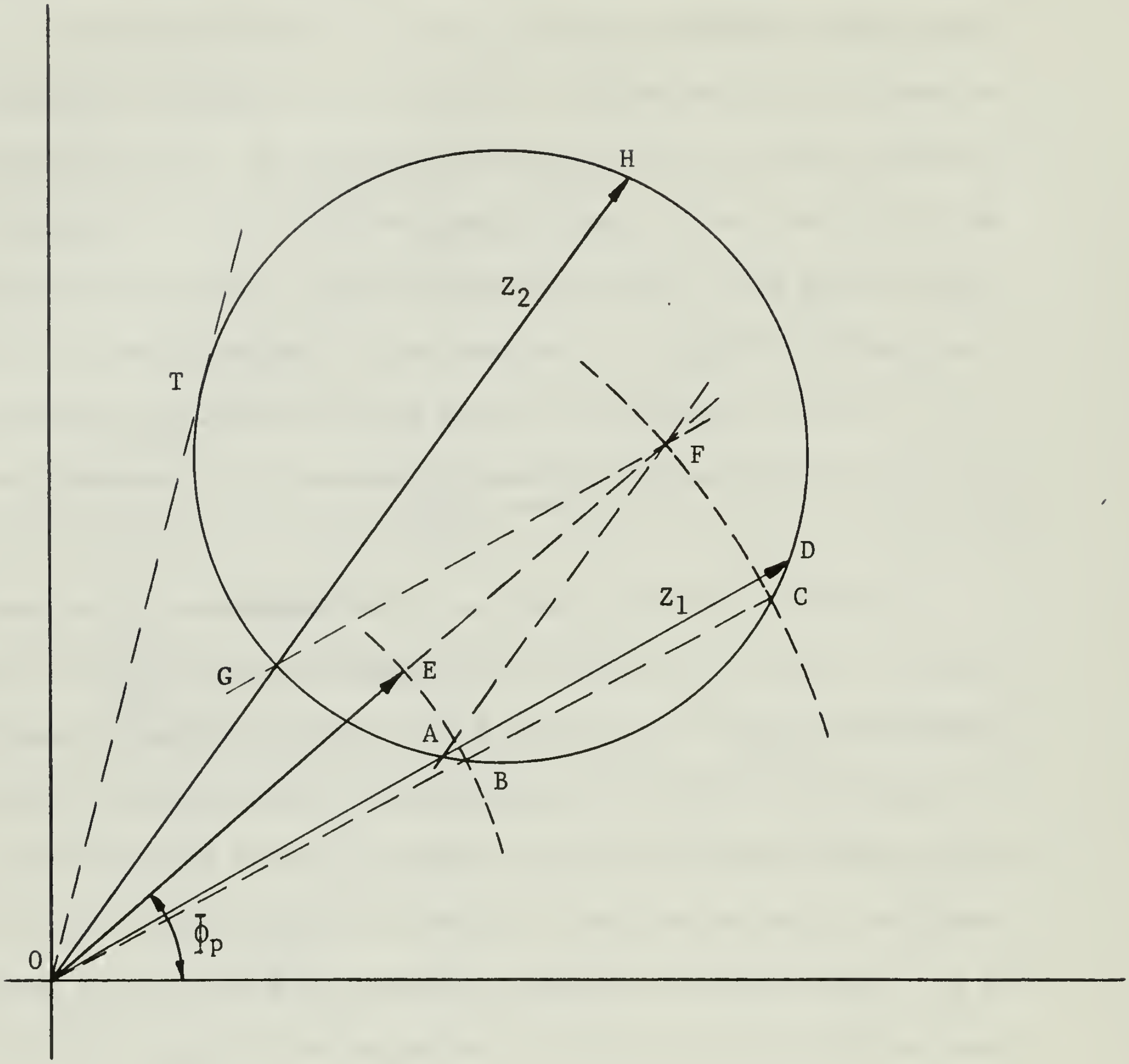


Figure 5-12. Arbitrary Impedances, Both in Same Quadrant

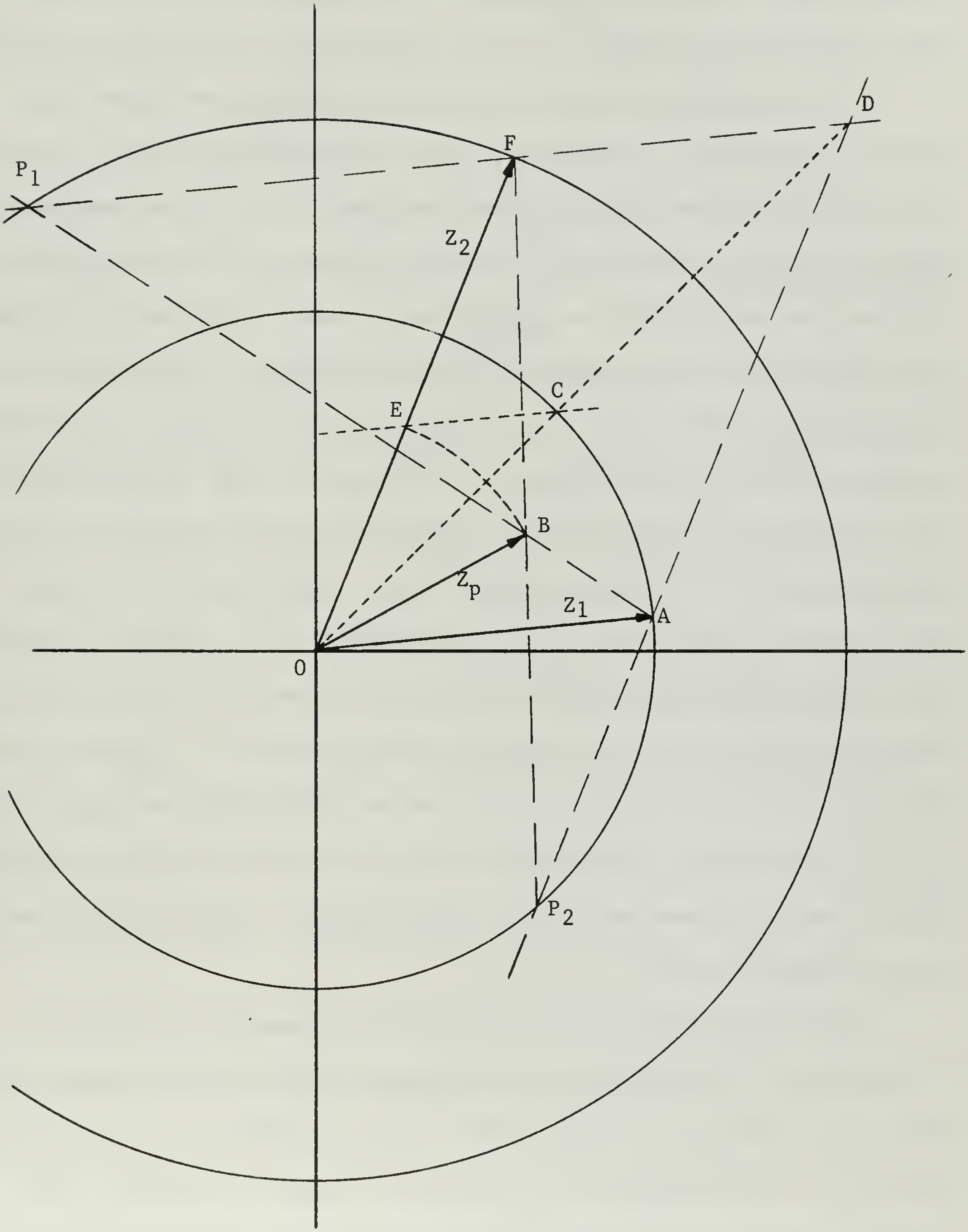


Figure 5-13. Arbitrary Impedances, Method by Storch, Requiring Special Graph Paper

the given impedances Z_1 and Z_2 are drawn. Z_1 and Z_2 are added vectorally to OD, thereby forming a parallelogram whose sides DF and DA are extended to get points P_1 and P_2 respectively. The equivalent impedance Z_p is now represented by OB, where B is the intersection of line AP_1 with FP_2.

In solving the problem of paralleling impedances one often first converts impedances to admittances, then adds those and finally inverts the total back to get the equivalent impedance. R. C. Paine developed a method relying on the theorem of geometry which states that the perpendicular from the vertex of a right triangle onto the hypotenuse is a mean proportional between the segments of the hypotenuse. The application is illustrated in Figure 5-14. Z_1 and Z_2 are known impedances connected in parallel. A circle with midpoint at O is drawn in such a manner that it passes between the tips of the two impedance vectors. Through O a perpendicular to OG is erected giving OE and a right angle at E is formed with line GE to give point B on the extension of Z_1. Similarly points F and D are found. By the above theorem the reciprocal of Z_1 is OB and the reciprocal of Z_2 is OD, both multiplied by a constant k. OB and OD are added vectorally to OC which is the inverse of the equivalent impedance Z_p. OC is inverted in the same way as it was already done above yielding point H, with OH representing Z_p. The constant k cancels in the course of the second inversion.

A very similar method to the one outlined above was developed by R. Panholzer. This method, too, progresses in four distinct steps that permit good insight into the problem. The four steps are normalization, inversion, vector addition, and inversion. Figure 5-15 depicts this method. A circle is drawn with center at O and passing through the tip of one of the given impedances Z_1 and Z_2. The radius of this circle, in

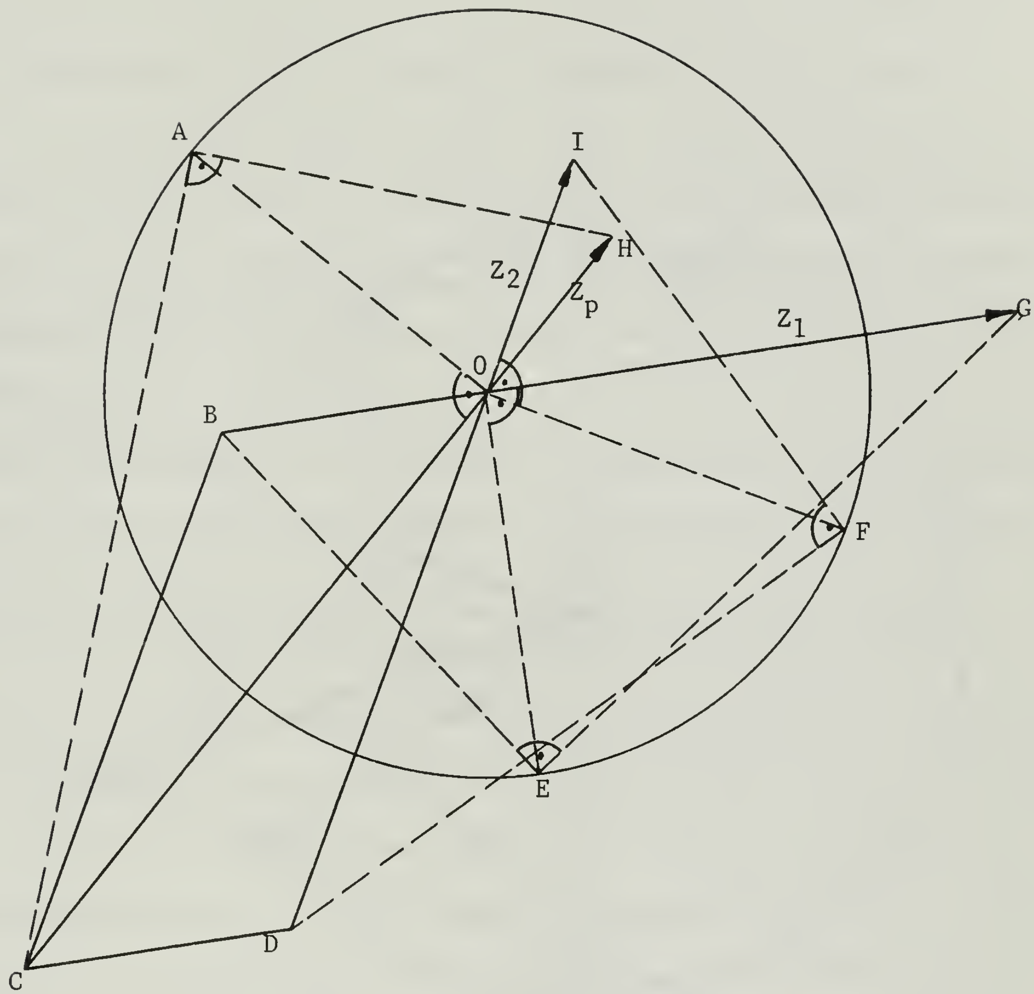


Figure 5-14. Arbitrary Impedances, Inversion Method

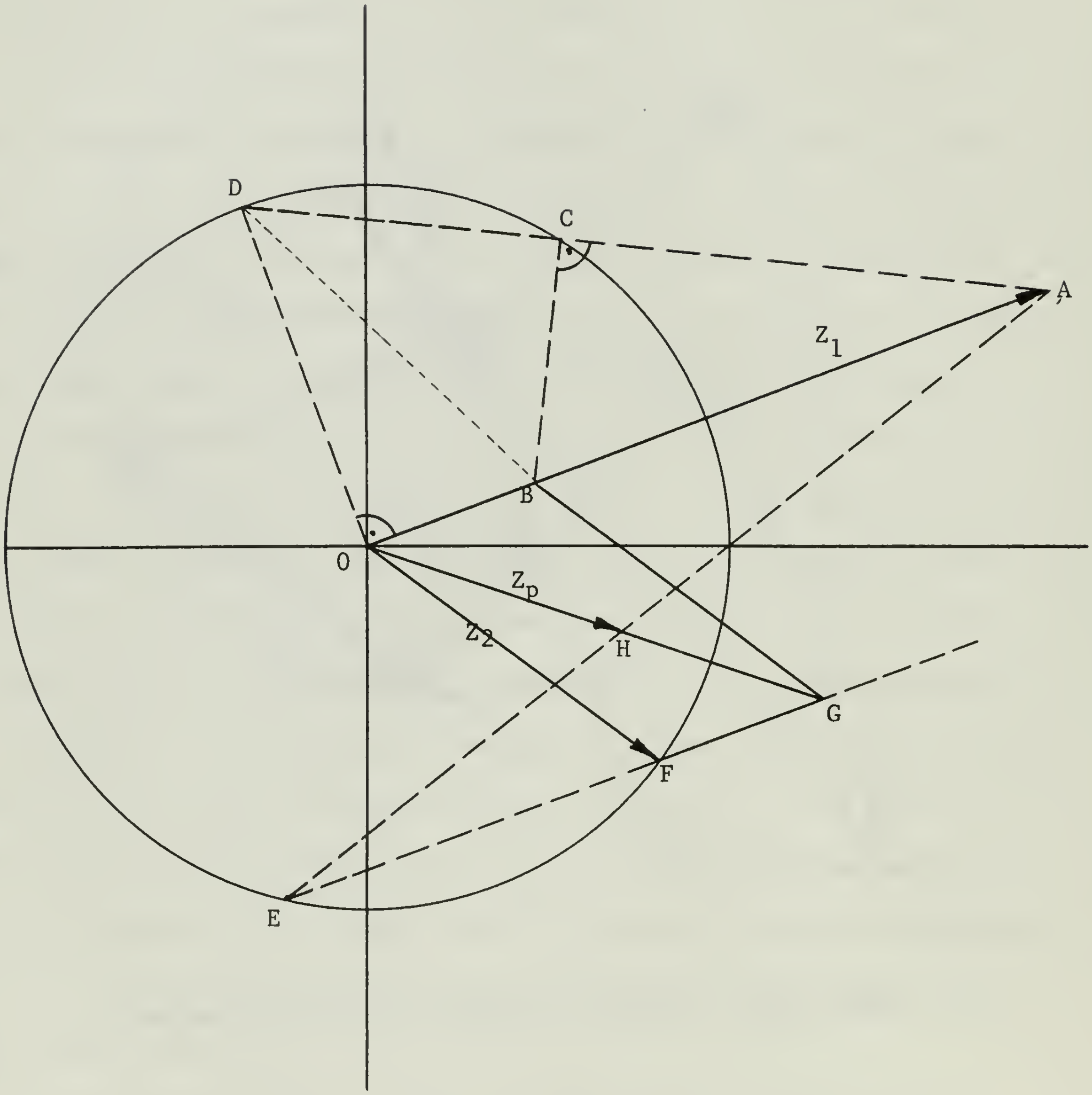
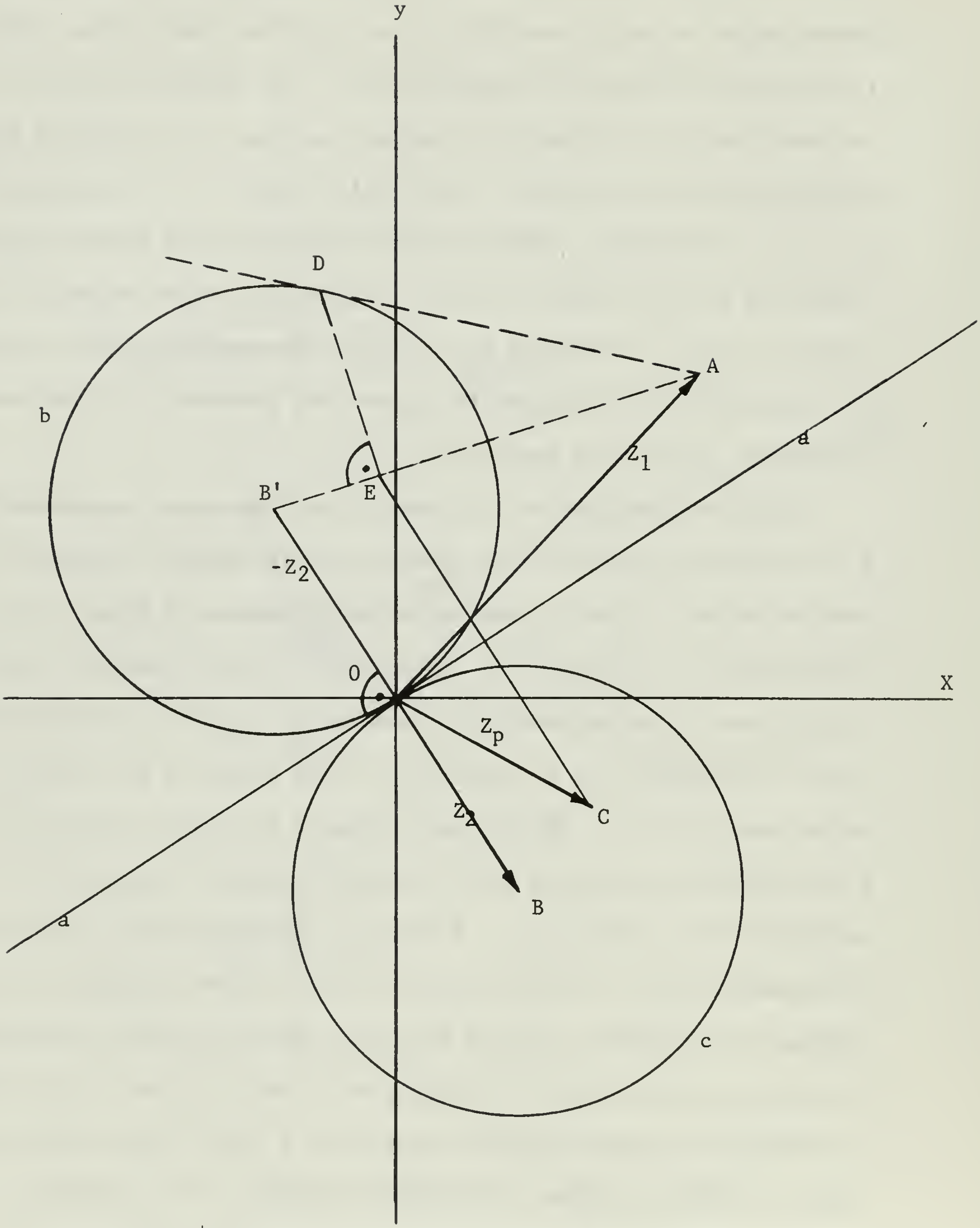


Figure 5-15. Arbitrary Impedances, Method by Panholzer

this case $|Z_2|$, is set equal to 1.0, thus normalizing the dimensions. A perpendicular to OA is erected at O and cuts the circle at D. Point C is obtained by finding the intersection of line AD with the circle. A perpendicular to AD through C is set up, cutting Z_1 at B with OB representing the inverse of $|Z_1|$. Since $|Z_2|$ is equal to 1 its inverse is OF, too. OF and OB are added vectorally to OG. OG is inverted again by connecting A with E, where E is the intersection of the extended line GF with the circle, and finding H. H is the intersection point of OG with AE. The phasor OH represents the equivalent impedance Z_p. The proof to this method is given in Appendix 17.

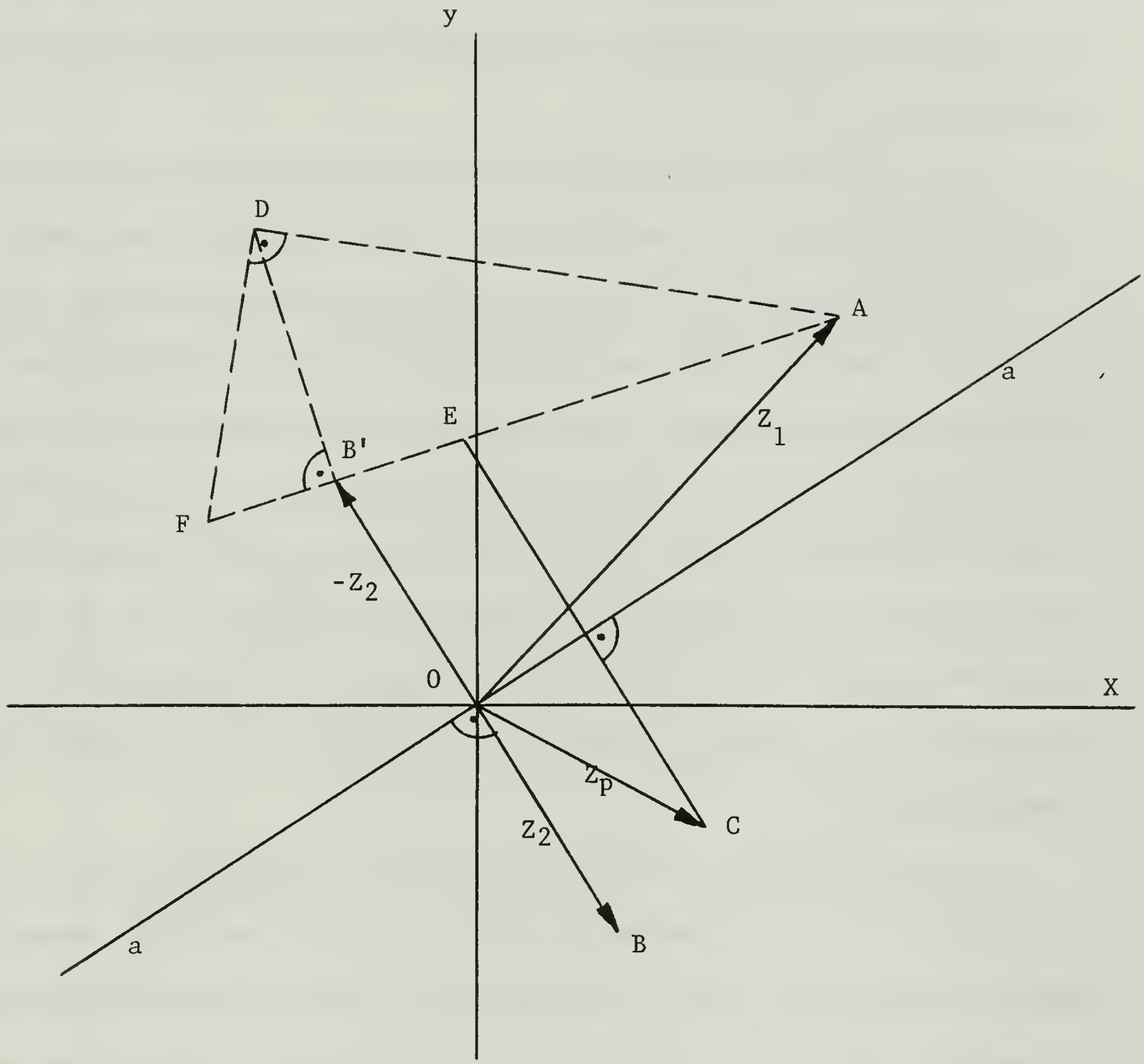
A somehow different way to parallel two impedances originates with E. F. Bolinder who applies the isometric circle method, a graphical inversion method. Z_1 and Z_2 are the given impedances in Figure 5-16. First point B', the tip of the $-Z_2$ phasor has to be located. Then two circles have to be sketched, both with radius $|Z_2|$ and passing through O, circle b with center at B' and circle c with center at B. Next Z_1 is inverted into circle b. The tangent through A to circle b fixes point D. A perpendicular erected to B'A and passing through D fixes point E. B'E represents the inverted $|Z_1|$. E is then reflected to the other side of the symmetry line a to give point C, where a goes through O and is perpendicular to line BB'. Phasor OC is the desired parallel equivalent Z_p. The same Z_p can be found by inverting the roles of Z_1 and Z_2 and drawing two circles with radius $|Z_1|$ and centers at A and A' but otherwise following the same procedure. The proof is worked out in Appendix 18.

The construction in Figure 5-17 avoids the use of isometric circles and can be done with straight lines only. This method, too, originates with E. F. Bolinder. Line AB' is drawn and extended beyond B'. The



$r_b = r_c = |Z_2|$

Figure 5-16. Arbitrary Impedances, Bolinder's Isometric Circle Method



$B'F = B'E$

$B'D = |Z_2|$

Figure 5-17. Arbitrary Impedances, Simplified Isometric Circle Method

perpendicular to line AB' at B' gives point D by letting $B'D = |Z_2|$. The perpendicular erected on AD at point D cuts AB' at F. Shifting the distance FB' to the right of B' fixes point E. E is then reflected about the symmetry line a to get C. The phasor OC is the desired equivalent impedance Z_p.

In the following three methods are presented which are advantageous if the angle between the two impedances is very obtuse or very acute.

The first one shown in Figure 5-18 was introduced by Boehne and it consists of the successive use of the series parallel conversion and thus provides good insight into the problem. The phasors Z_1 and Z_2 have to be reduced to their parallel parameters R_{p1}, X_{p1} and R_{p2}, X_{p2} respectively, according to Figure 1-1. The corresponding like parameters are then paralleled according to the construction in Figure 2-2. R_p represents the parallel equivalent of R_{p1} and R_{p2} and X_p represents the equivalent of X_{p1} and X_{p2}. The total impedance of R_p and X_p gives the desired value Z_p.

The second method is credited to E. Wilkinson and, as can be seen from Figure 5-19, its construction does not require any angle measurements but bisections have to be carried out. Given are the impedances Z_1 and Z_2. The tips of Z_1 and Z_2 are connected by a straight line and this line AB is bisected giving point J. Line OJ is extended on both sides. Points M and N are fixed by bisecting OA and OB respectively. The perpendicular to OA in point M cuts OJ in point L and the perpendicular to OB in point N cuts OJ in point K. Now lines KB and AL are drawn and their intersection fixes C. The phasor OC represents Z_p in magnitude and phase. This method becomes particularly compact if the phase angle between Z_1 and Z_2 is very acute. The construction of Figure 5-20

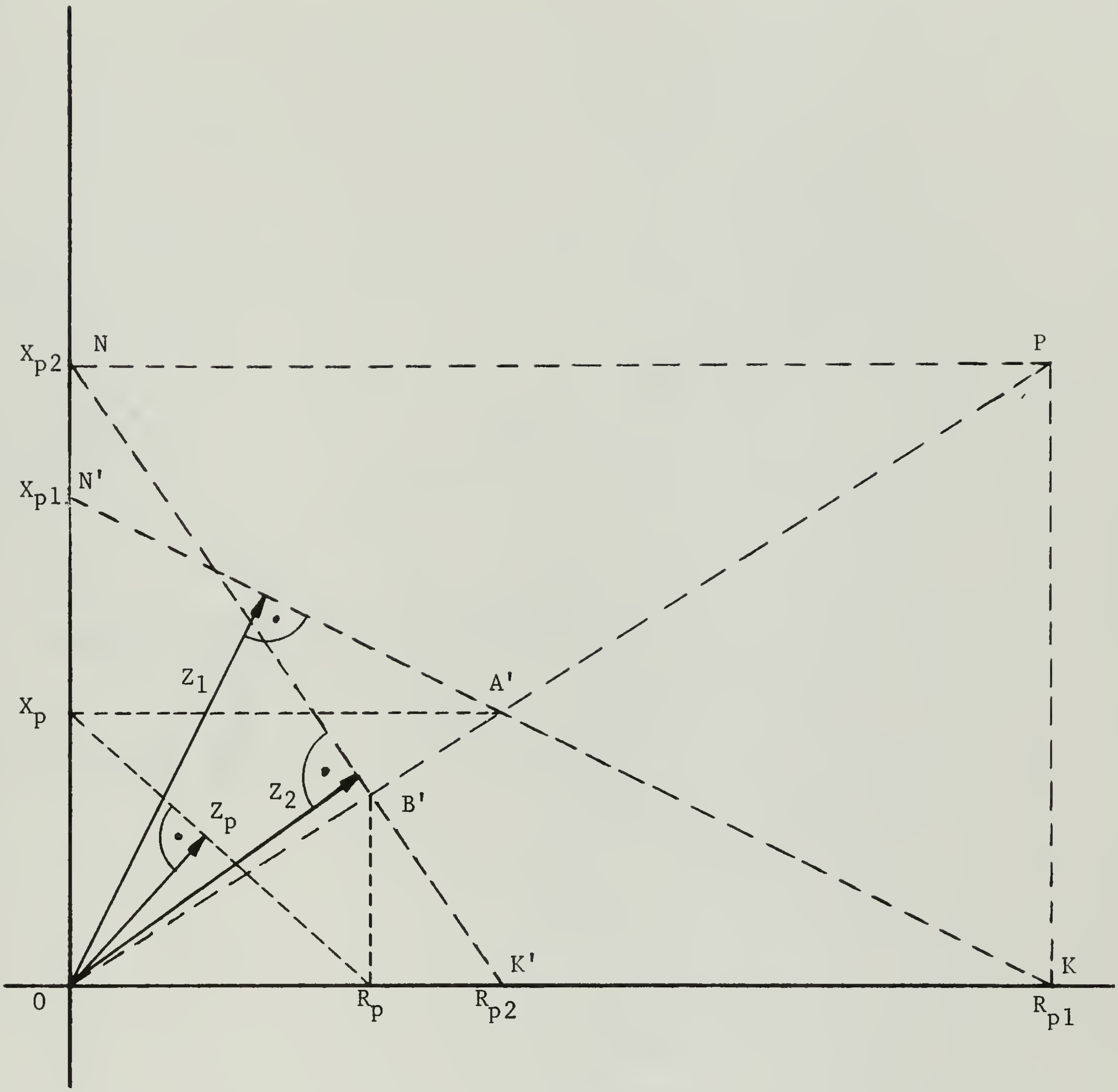
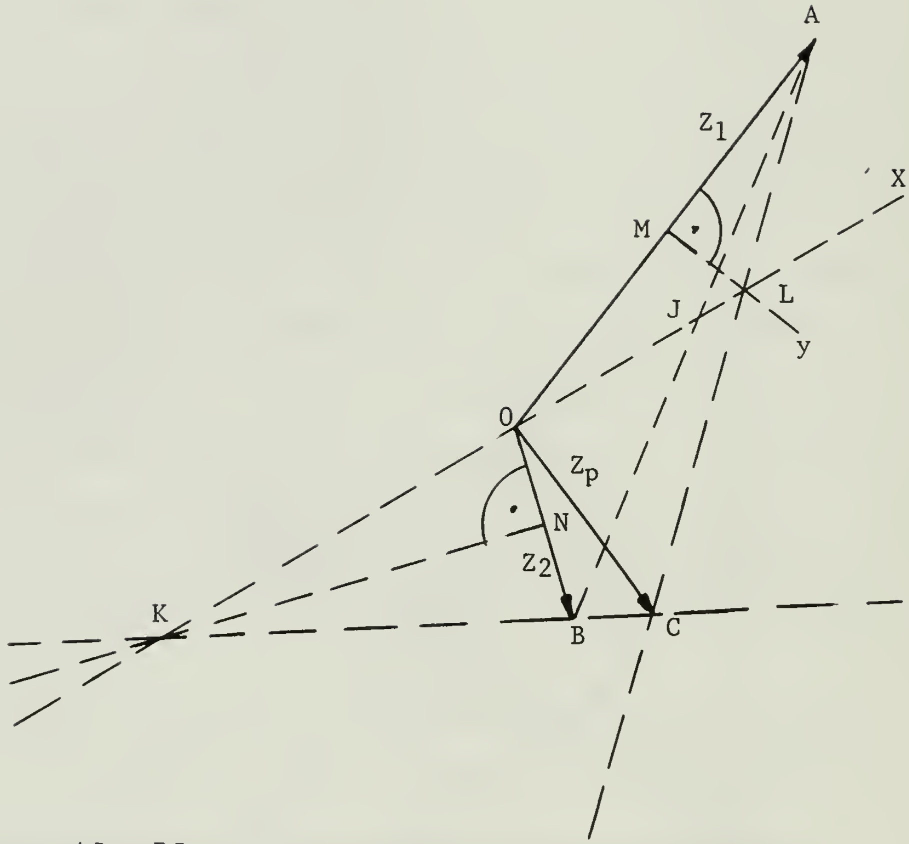


Figure 5-18. Arbitrary Impedances, Separated by Small Phase Angle



AJ = BJ

OM = AM

ON = BN

Figure 5-19. Arbitrary Impedances, Wilkinson's Method for Large Phase Angles

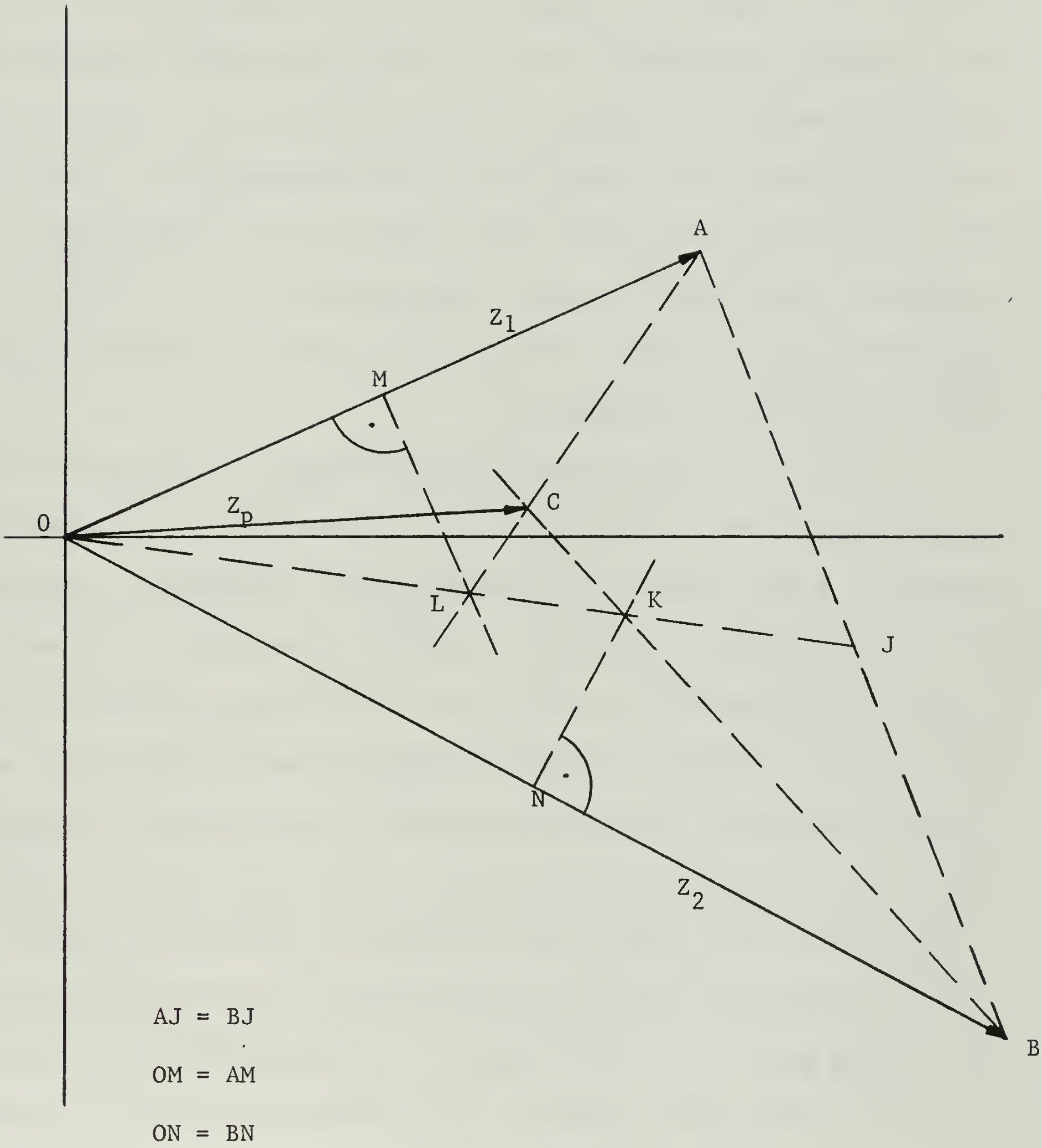


Figure 5-20. Arbitrary Impedances, Wilkinson's Method for Small Phase Angles

is the same as the one just described, just that the whole graphical work is executed inside the triangle OAB.

The third method for very obtuse or very acute angles was developed by Kind and is depicted in Figure 5-21. Two circles are drawn centered at O and going through the tips of Z_1 and Z_2 respectively, where Z_1 and Z_2 are the two given impedances. Z_1 is extended to form point D. Then line AD represents the vector $Z_1 + Z_2$. Now triangle ODA is rotated until its side DA is parallel to ED thus forming the new triangle OCB. A parallel to OB is drawn through E cutting line OC in F. The phasor OF represents Z_p, the parallel equivalent of Z_1 and Z_2. The proof of this method is laid out in Appendix 20.

At the end of this chapter some special cases to the general methods presented up to here are outlined. One of them developed by Stubbings applies to a pure resistance in parallel with an impedance. In Figure 5-22 R is the given resistance, Z the given impedance. R is laid off on the real axis and Z is drawn in such a way that its origin lies in the tip of R. Then OB is sketched forming the angle θ_s with the x-axis. A circle centered at O and passing through A, the tip of R, intersects OB at D. The distance AB is laid off on the y-axis to give OC. Line CB is drawn next and a parallel to it through D, fixing point E on the y-axis. OE represents then the magnitude of Z_p, while the phase angle θ_p of Z_p is given by the angle OBA. It can be seen that this method can be used to parallel two impedances with a phase difference θ. Proof of this method is given in Appendix 21.

For the same case of a resistance in parallel with an impedance Reppisch proposes a method based on the secants and tangents theorem of geometry. In Figure 5-23 R and Z are given in a cartesian coordinate

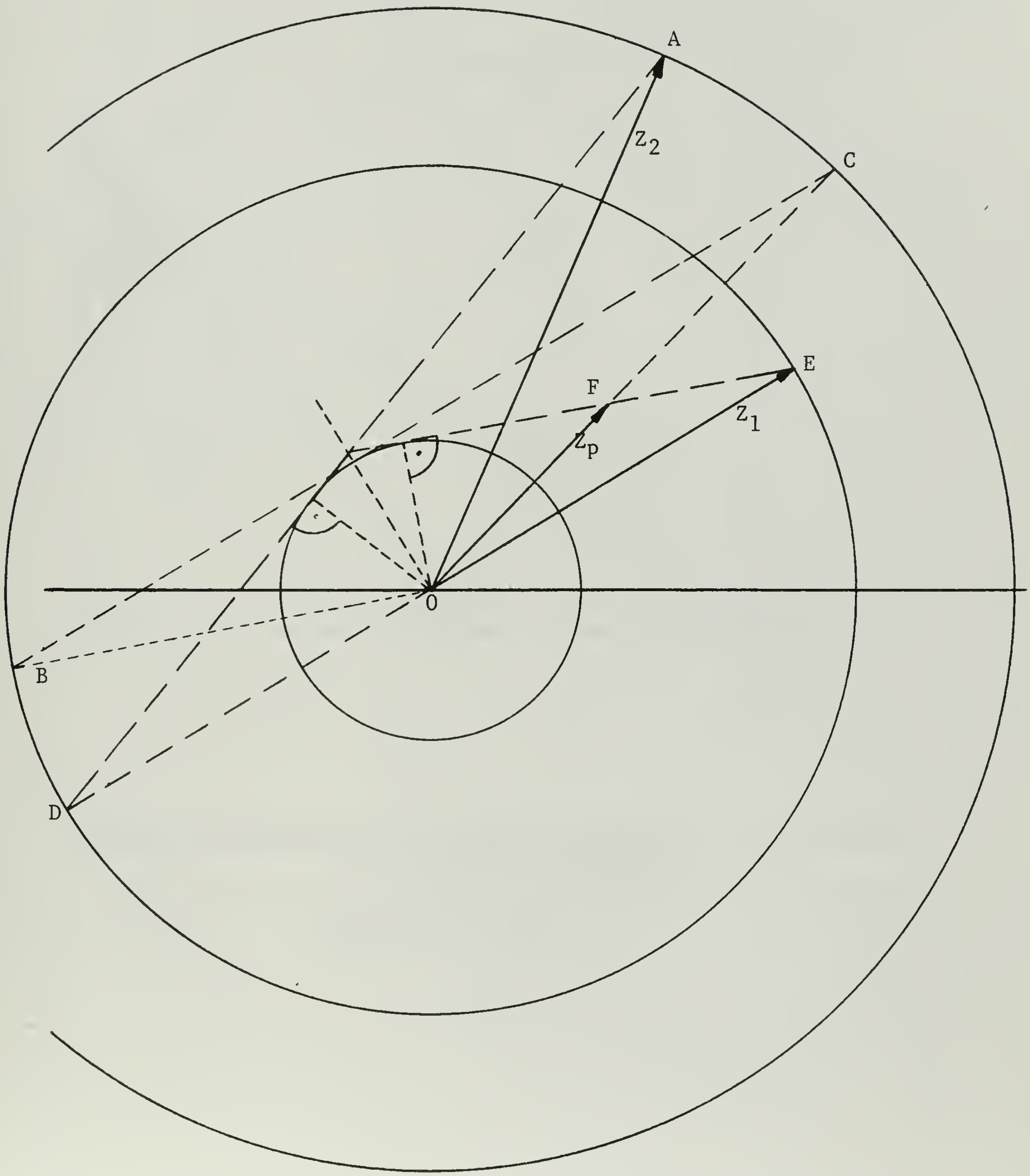


Figure 5-21. Arbitrary Impedances, Kind's Method for Acute and Obtuse Phase Angles

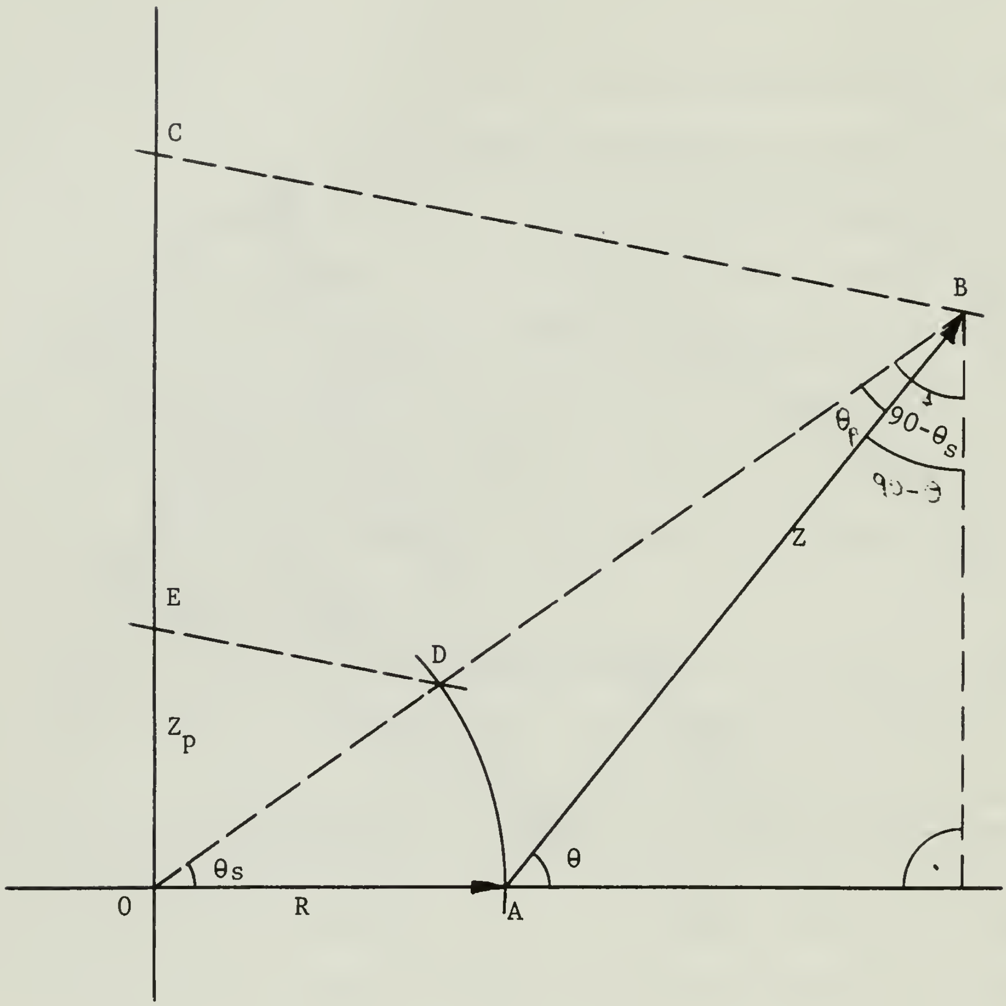


Figure 5-22. Resistance in Parallel With Arbitrary Impedance, Developed by Stubbings

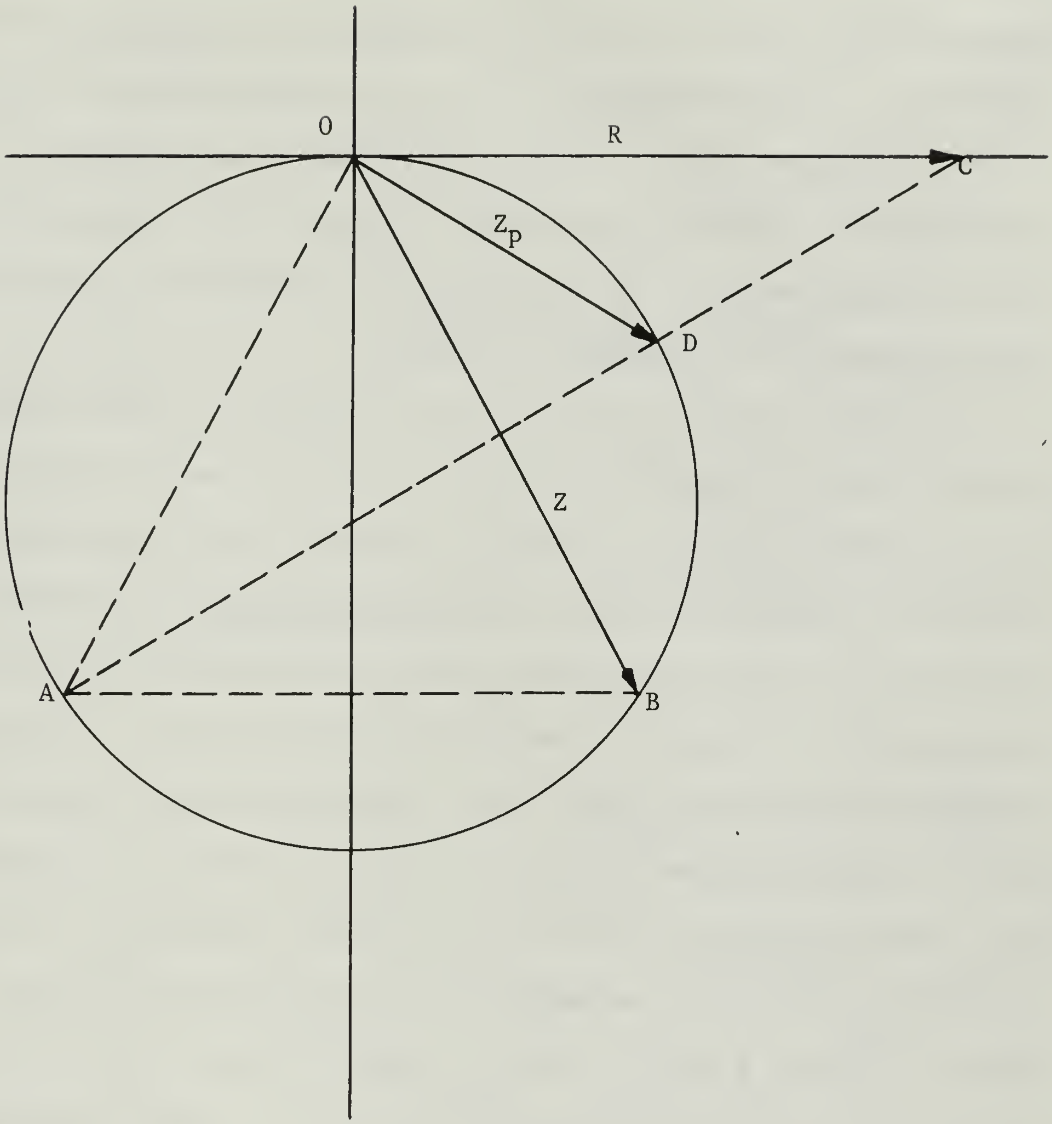
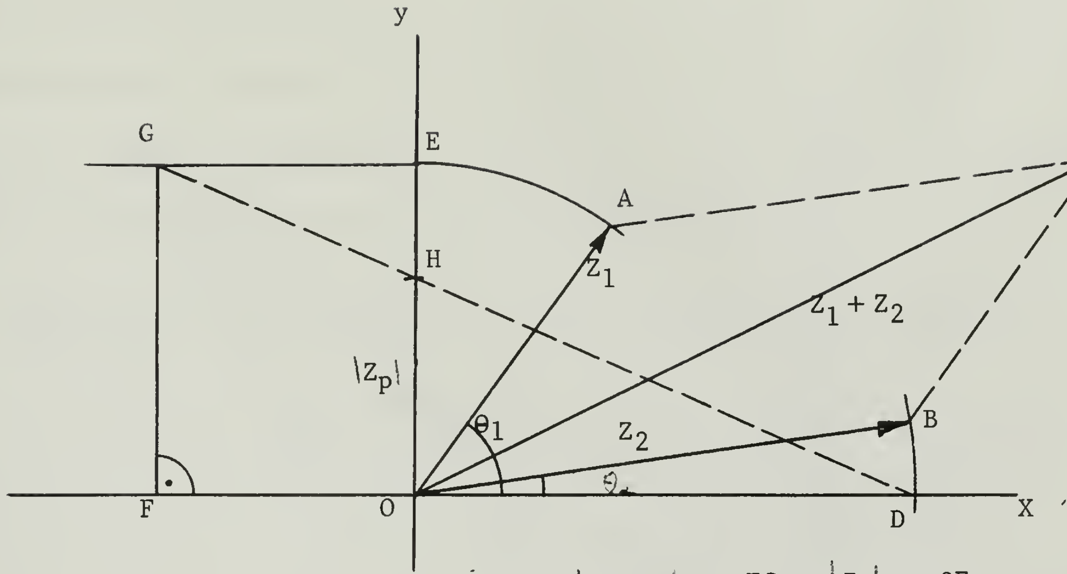


Figure 5-23. Resistance in Parallel With Arbitrary Impedance, Method by Reppisch

system and the phase angle of Z is assumed in the fourth quadrant, though the method works equally well for phase angles of Z which lie in the other quadrants. A circle tangent to R at O and passing through B, the tip of Z, has to be drawn. OB is reflected about the y-axis forming OA. A is connected with C and the intersection with the circle yields point D. Phasor OD represents Z_p the desired parallel impedance. Appendix 22 contains the proof to this method.

In Figure 5-24 N. H. Barker presents a method to obtain the magnitude of the parallel equivalent Z_p only. First OC is formed by adding the given impedances Z_1 and Z_2 vectorally. Points E and D are fixed by transferring the distance OA to the y-axis and the distance OB to the x-axis respectively. From point D the distance OC is measured off to the left fixing point F. Through F a perpendicular to the x-axis is erected giving point G, where FG is equal to OE. The connection GD crosses the y-axis at H and OH represents $|Z_p|$, the parallel equivalent in magnitude only. To get the phase angle of Z_p, too, the construction is extended to that of Figure 5-25, becoming rather complicated and laborious. In Figure 5-25 point H and thus $|Z_p|$ is found in exactly the same way as just described. Then the parallelogram OB'C'A' is formed by flipping over the parallelogram OACB. Line OC' has now the direction θ_p, and OH measured off on line OC' results in the phasor OH', representing Z_p in magnitude and phase. The proof is worked out in Appendix 22.

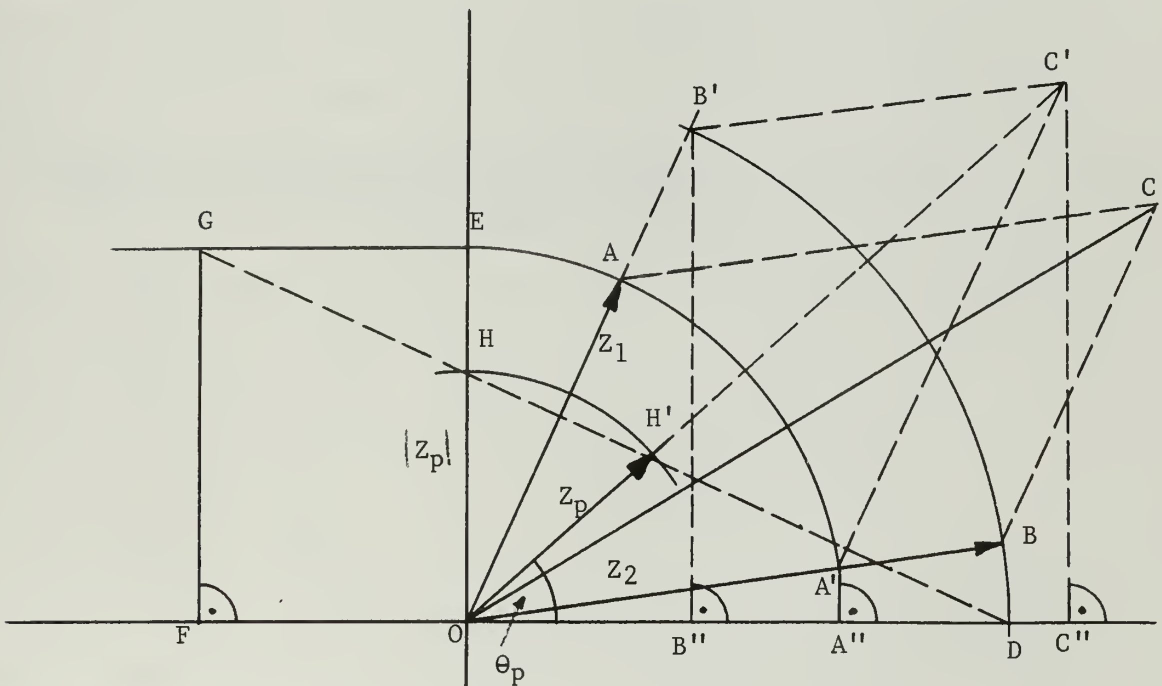
Paine and Stubbings use the ray theorem for another method that only provides the magnitude of the equivalent impedance. Figure 5-26 pictures the construction. Z_1 and Z_2 are the given impedances. They are drawn with their correct relative phase difference, but one of them has to coincide with one of the coordinate axes, here Z_1 coincides with the x-axis.



$FD = |Z + Z_2|$; $FG = |Z_1| = OE$

$OD - |Z_2|$; $OH = |Z_p|$

Figure 5-24. Arbitrary Impedances, Method by Barker, Yielding Magnitude Only



B'K // OA'

A'K // OB'

$|OC| = |OK|$

Figure 5-25. Arbitrary Impedances, Method by Barker, Yielding Magnitude and Phase

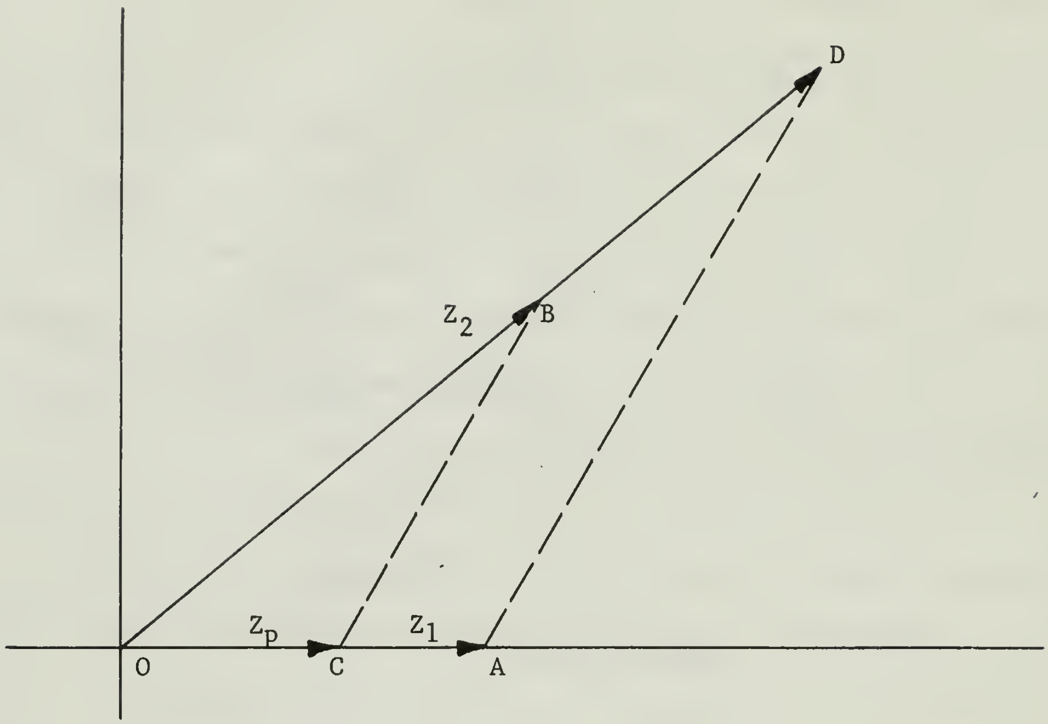


Figure 5-26. Arbitrary Impedances, Applying Ray Theorem, Yielding Magnitude Only

OA is measured off from B on the extended line OB giving OD. Line AD is sketched and the parallel to AD intersects the x-axis at C. OC then represents $|Z_p|$, the magnitude of the parallel equivalent of the impedances Z_1 and Z_2. The proof is outlined in Appendix 23.

6. Reduction of a Bridge Network of Like Impedances to its Equivalent Impedance.

The problem of finding graphically the total impedance of a bridge network consisting of like impedances as shown in Figure 6-1 was first solved by Neumann as early as in 1914. Later Rukop republished the method in a slightly different form. Another method was described by Rauschenberg some 30 years ago, resorting to the geometrical analogy of a divergent or convergent infinite series. The principles underlying these constructions are shortly described in the following: Assuming that the emf of the source and the five impedances of the branches are given and using a rectangular coordinate system in which currents are drawn parallel to the ordinate, voltages parallel to the abscissa, impedance values then appear as cotangents of angles. If $E_1 \ldots\ldots E_5$ are voltage drops across the respective branches and $I_1 \ldots\ldots I_5$ are currents through these branches, then:

$Z_1 = E_1/I_1 = \text{ctg } \varphi$ $\qquad$ $Z_2 = E_2/I_2 = \text{ctg } \epsilon$

$Z_3 = E_3/I_3 = \text{ctg } \lambda$ $\qquad$ $Z_4 = E_4/I_4 = \text{ctg } \psi$

$Z_5 = E_5/I_5 = \text{ctg } \eta$

The method is now being applied to find the total impedance of a bridge network like the one depicted in Figure 6-1. Figure 6-2 shows the construction. Let emf E across the terminals of the total net and the five branch impedances be given in value. First Z_5 is represented by a rectangle of optional size but with ratio of width to height

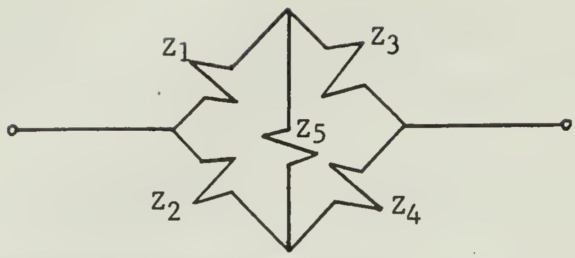


Figure 6-1. Bridge Network
of Like Impedances

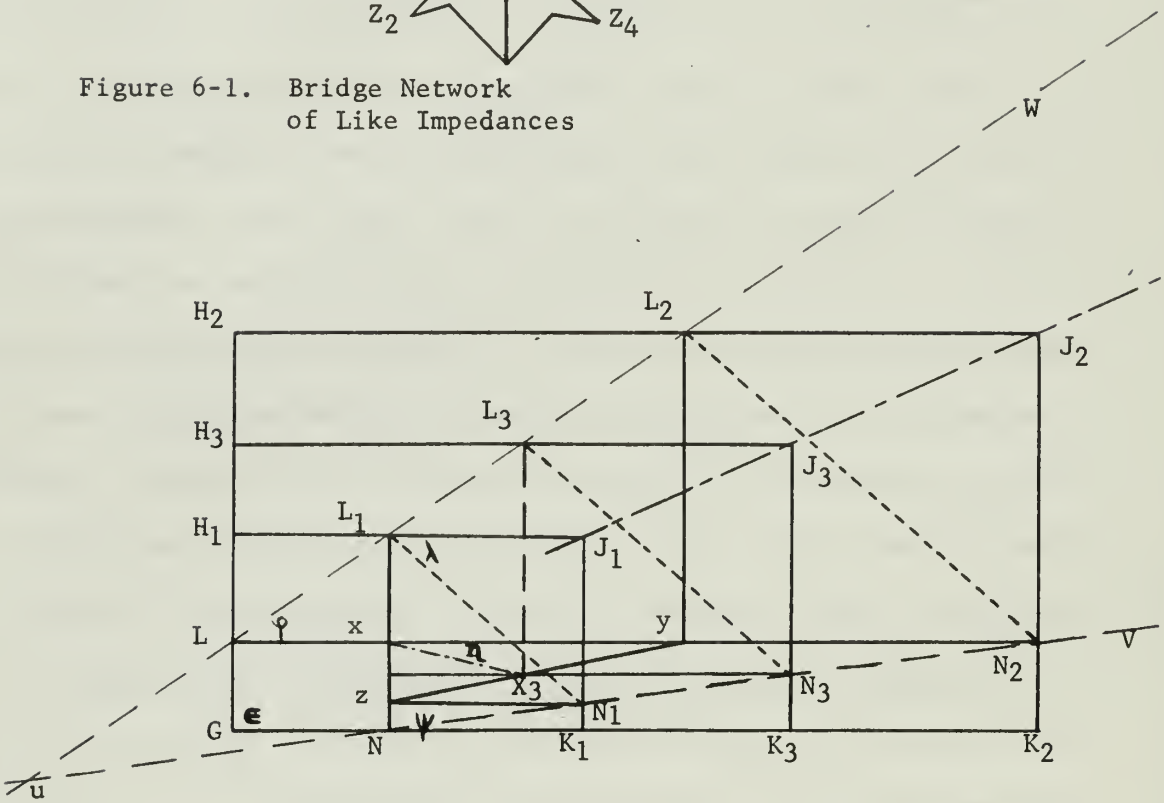


Figure 6-2. Neumann's Method for Finding Equivalent Impedance
of Bridge Network

equalling $Z_5 = \text{ctg}\,\eta$. This is GNXL. The two bordering branches Z_4 and Z_1 are represented by angles ψ and φ drawn in through N and L respectively, where the cotangents of these angles again represent the ratio of abscissa to ordinate, i.e. they are measured from the ordinate down towards the abscissa. Impedances Z_1 and Z_4 provide two rays W and V which intersect in U. Using Z_3, represented by the cotangent of λ, the two limiting cases can be constructed: the bridge is open (no current flows, the "no-load-voltage" is obtained), the bridge is short-circuited (no voltage drop across Z_5, the "short-circuit-current" is obtained). The first case furnishes rectangle $GK_2J_2H_2$, XY representing the "no-load-voltage", the latter case gives rectangle $GK_1J_1H_1$, XZ representing the "short-circuit-current". All cases in which Z_5 has a finite value form a small rectangle whose four corners lie in X, on XZ, XY, YZ. As an example Z_5 is represented by the cotangent of η. The diagonal at angle η is drawn starting at X, intersecting ZY at X_3 which leads to points L_3 and N_3, where line L_3N_3 is parallel to lines L_1N_1 and L_2N_2 was found before when the two extreme cases were constructed. The total impedance is gotten from rectangle $GK_3J_3H_3$ by measuring the angle between the ordinate and the diagonal:

$$\text{ctg}\,\eta = E/I = GK_3/GH_3 = Z$$

The scale of the diagram is finally found by equalling GH_3 to E which is given and all other values can be derived therefrom. It can be shown that an extension of line YZ will go through point U, so does the line connecting points J_2, J_3, J_1. Furthermore triangles L_1ZN_1, L_2YN_2, and $L_3X_3N_3$ are similar.

A second method using the asymptotic approach of a converging or diverging infinite series is depicted in Figure 6-3. It is again assumed

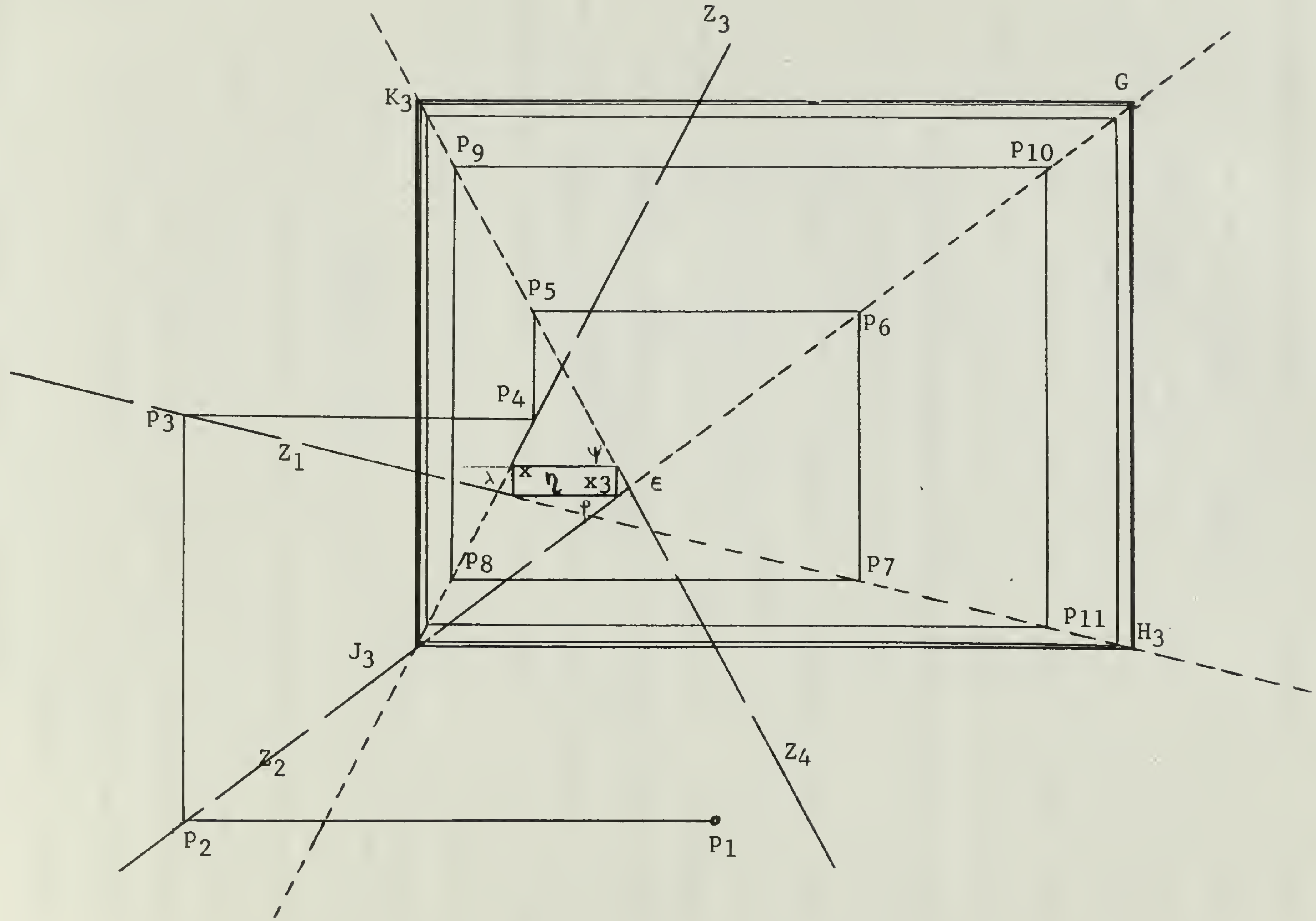


Figure 6-3. Rauschenberg's Method for Finding Equivalent Impedance of Bridge Network

that all five like impedances are known, together with the voltage across the input terminals, E.

Let Z_5 be represented by a rectangle of any size whose ratio of width to height equals $Z_5 = \mathrm{ctg}\,\eta$, in this case XX_3. Now the angles ϵ, ψ, λ, φ are drawn from the corners of this rectangle, as shown in Figure 6-3, forming the impedance lines $Z_1 \ldots\ldots\ldots Z_4$. Later these lines will turn out to be the diagonals of the bordering rectangles as can be seen on the graph. The lines are sketched to both sides from those corners differing in a different way. To find the total impedance rectangle one starts from some arbitrary point, here P_1, drawing a parallel to the abscissa of the rectangle with diagonal XX_3. This parallel intersects the Z_2-line in P_2, then the Z_1-line is intersected by the second parallel, this one parallel to the ordinate of the original rectangle, in point P_2, the Z_1-line in turn is cut in P_3, and the Z_3-line in P_4. Here a switch is made to the differently marked end of the impedance line Z_4 thus producing a 180° rotation of the corners of the construction. In case the line diverges one has to go back and start to the other side, now obtaining a converging rectangle in 15 to 20 lines. In any case the end result is a rectangle, $GK_3J_3H_3$, the cotangent of the angle between the ordinate and the diagonal of which results in Z_{total}. The scale again is fixed when GH_3 is set equal to the given emf E, all voltages and currents can then be read off according to that scale.

PART II: GRAPHICAL METHODS IN NETWORK SYNTHESIS

7. Simple Matching Networks for Maximum Power Transfer.

In this chapter several methods for synthesizing L, Pi or T-type matching networks are developed. In the course of these constructions often inversion or paralleling problems are encountered and solved with methods discussed in foregoing chapters. The first three sections of this chapter look into methods concerned with L matching sections only. L sections are the cheapest and easiest ones to implement; however, they do not provide as much flexibility as T and Pi sections do. This will be demonstrated in the fourth section, where methods to synthesize T and Pi sections with the help of voltage and current vectors only are described. The methods introduced in this chapter all follow a clear step by step procedure that greatly facilitates insight into and understanding of impedance matching.

Before going into the first method some general thoughts and definitions are put down. The purpose of a matching network is to insure maximum power transfer from some source with complex impedance Z_s to some load with complex impedance Z_l. Maximum power transfer occurs if the load "sees" the source impedance Z_s as Z_l^*, the conjugate of Z_l, or if the source "sees" the load impedance Z_l as Z_s^*, the conjugate of Z_s. To change the corresponding impedance to the desired value only reactive components are used to keep the matching network lossless. There exist two types of L-sections that can be used, the A-type pictured in Figure 7-1 and the B-type pictured in Figure 7-2. In some matching problems B as well as A sections will do it, but sometimes only either type is applicable.

R. Panholzer developed a simple method to determine which type

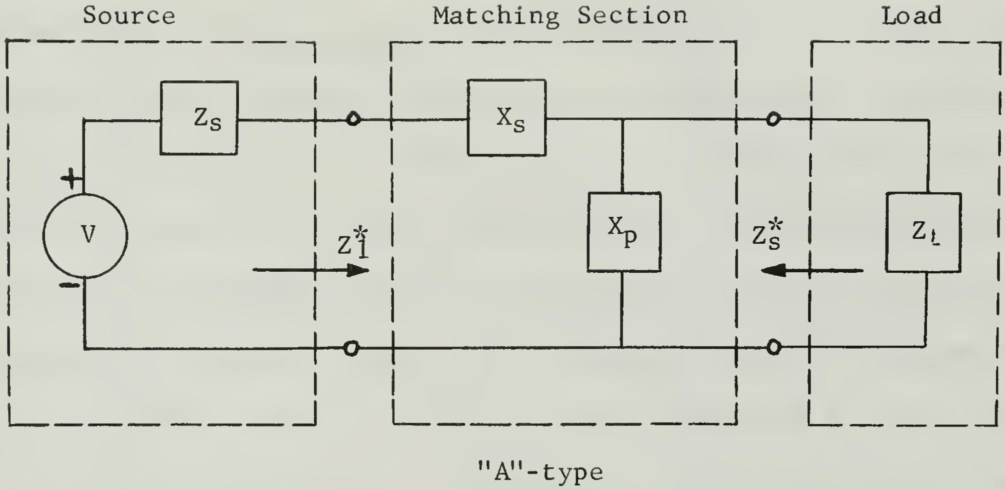


Figure 7-1. Source-Load Arrangement With Matching Network, "A"-Type L-Section

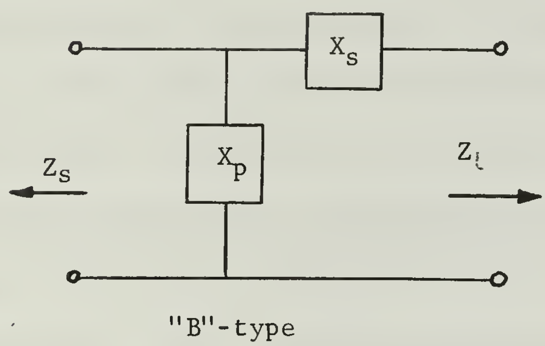


Figure 7-2. "B"-Type L-Section

L-section has to be used. The construction is presented in Figure 7-3. For a given load Z_1 a circle is drawn centered on the x-axis, tangent to the y-axis at 0 and passing through A, the tip of the load impedance vector Z_1. This circle cuts the x-axis at point C and a perpendicular to the x-axis, line a. is erected there. Then circle b is drawn, centered on the x-axis, tangent to the y-axis at 0 and going through B, where OB is the real part of Z_1. The source impedance Z_s is needed to determine which type L-section can be used. If the tip of Z_s falls inside circle b then only A-type, if it falls to the right of line a, then only B-type can be used. Either A or B-type will do the job if the tip of Z_s falls into the region outside of circle b and to the left of line a.

The first section consists of two methods proposed by R. Panholzer. In the first method the matching is accomplished with an A-type L-section. Figure 7-4 depicts this construction. For the given Z_s and Z_1 either A- or B-type can be used, as the tip of Z_s falls outside of circle b and to the left of line a. For demonstration purpose an A-type L-section will be implemented. To find the value of X_s, the series reactance in the L network, a line perpendicular to the x-axis is drawn, going through C and D, the tips of Z_s^* and Z_s respectively. This line cuts circle b at B and E. Phasor BC, representing a capacitance, is one possible value for X_s, the other one is phasor EC, a inductance. To get the shunt reactance X_p of the L network points A' and B' are needed. They are located at the intersections of the extended lines OA and OB with line a. The phasor B'A' is the inverse of X_p. To invert, B'A' is shifted into the position B" A". Line OB" cuts circle b at P and the extended line A"P produces P' in the y-axis. OP represents one possible value of X_p, a capacitance. OT represents another possible value of X_p, this time an

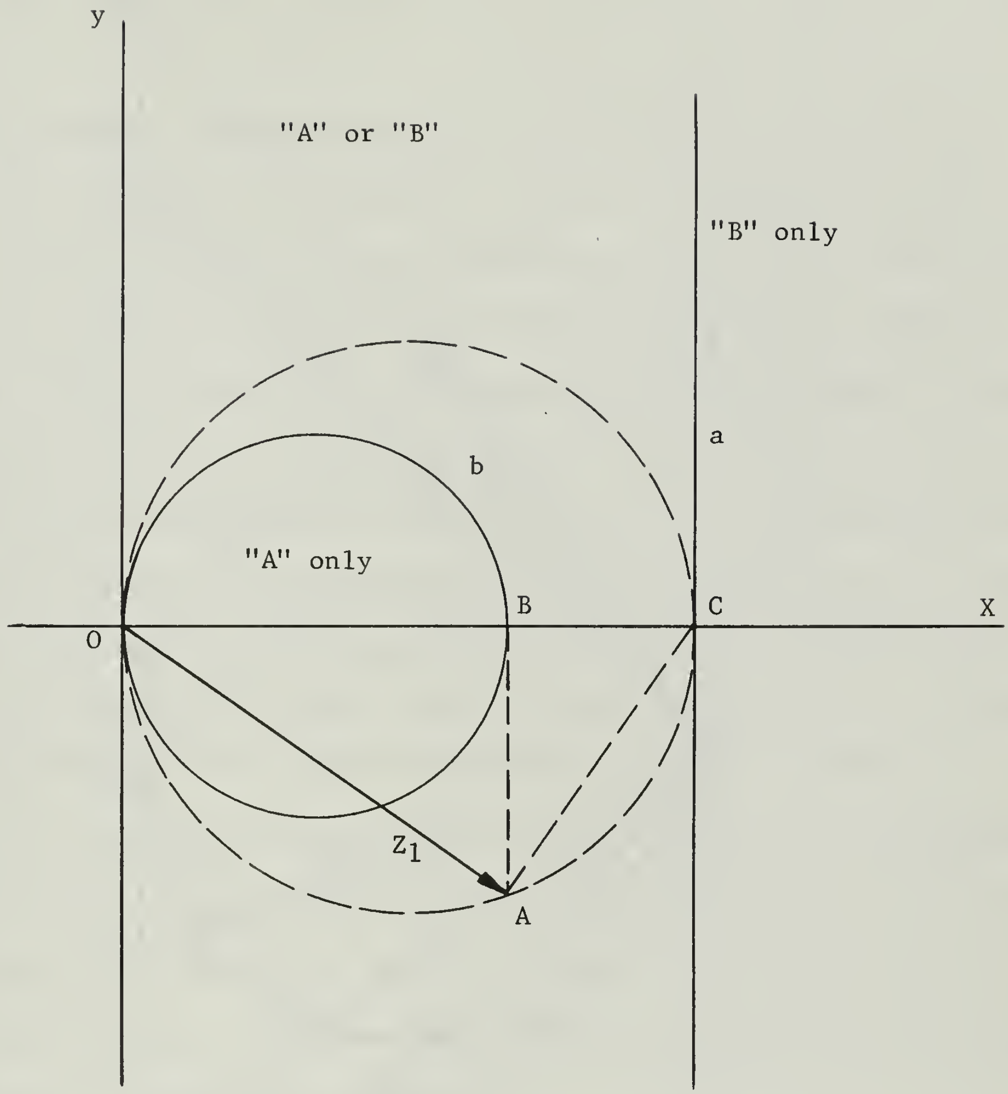


Figure 7-3. Determination of Type of L-Section to be Used

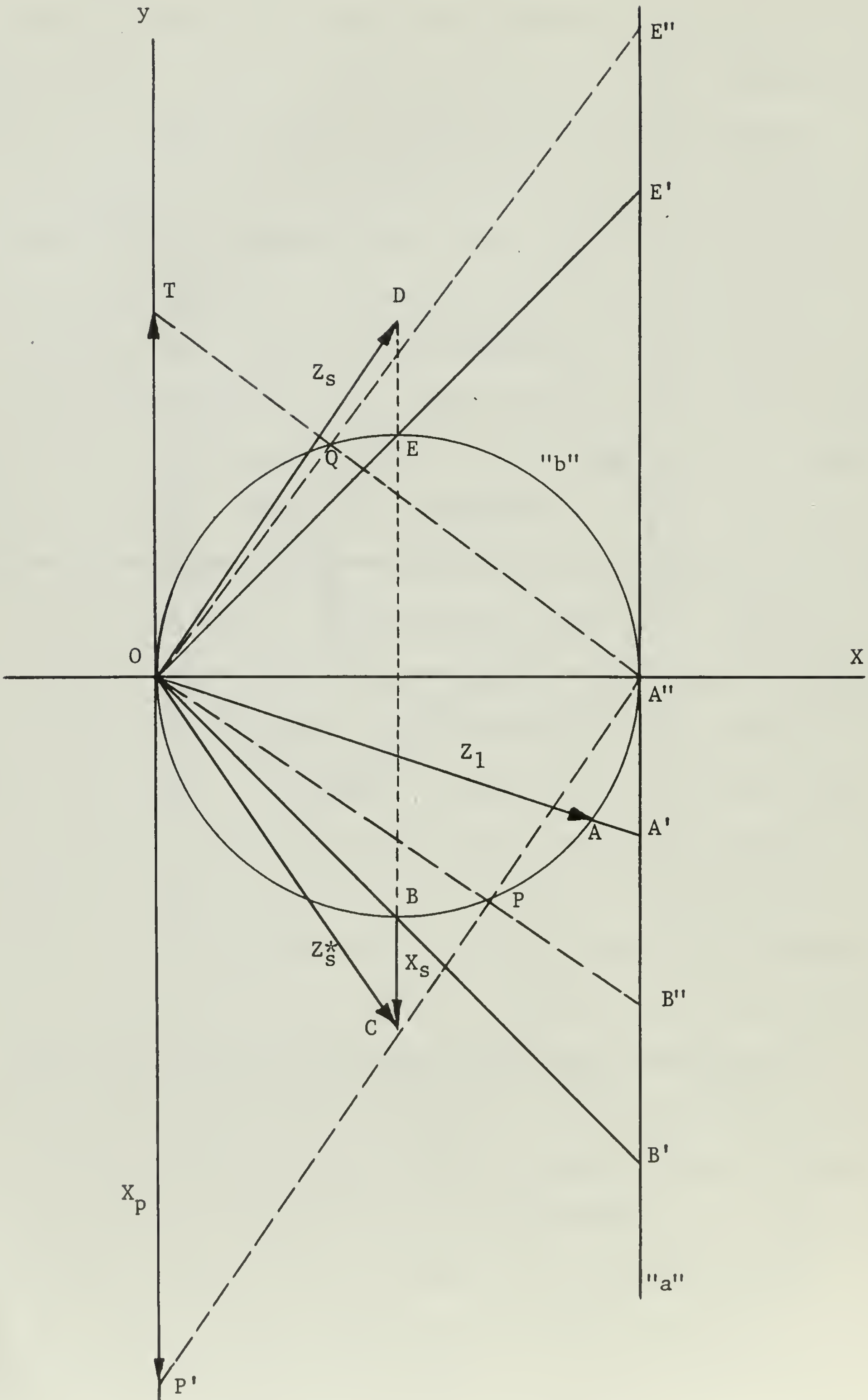


Figure 7-4. Example of Impedance Matching With "A"-Type L-Section

inductance. OT is found similarly to OP'.

The second method demonstrates the construction leading to a B-type L-section as shown in Figure 7-5. To find X_s a perpendicular to the x-axis is drawn, passing through point A, the tip of the Z_1 vector. This line cuts circle c in B and E. Phasor AB represents one possible value for X_s, a capacitance, while phasor AD represents the second possibility for X_s, an inductance. To find X_p first points B' and C' are needed. They are fixed by the intersections of the extensions of lines OB and OC with line a. Phasor C'B' has to be inverted to yield X_p. To invert, C'B, is shifted into the position C" B". O is connected with C" cutting circle c at P and the extended line KP produces point P' on the y-axis. OP' represents one possible X_p, a capacitance. The other possible X_p is found in a similar fashion by shifting phasor C'D' into the position C'''D'''.

The second section deals with a technique published by J. Deignan. The principle on which this technique is based has already been put into use earlier by L. Batchelder and R. C. Paine, too, has applied it to the same kind of problem. This method of matching a load to a source will be applied to three different cases.

First an L network is designed to match the load R_1 to the source R_s. In other words R_1 has to appear to the generator as R_{in}, where $R_{in} = R_s$. This construction is presented in Figure 7-6. To achieve this "change" of R_1 a reactance X_p has to be put in parallel with R_1 and then another reactance X_s has to be added in series to cancel the reactive component of Z_{Υ}, the equivalent of R_1 and X_p in parallel. From Figure 7-3 it can be seen that only an A-type L-section can achieve this goal. R_1 is laid off on the real axis. A circle is drawn centered on

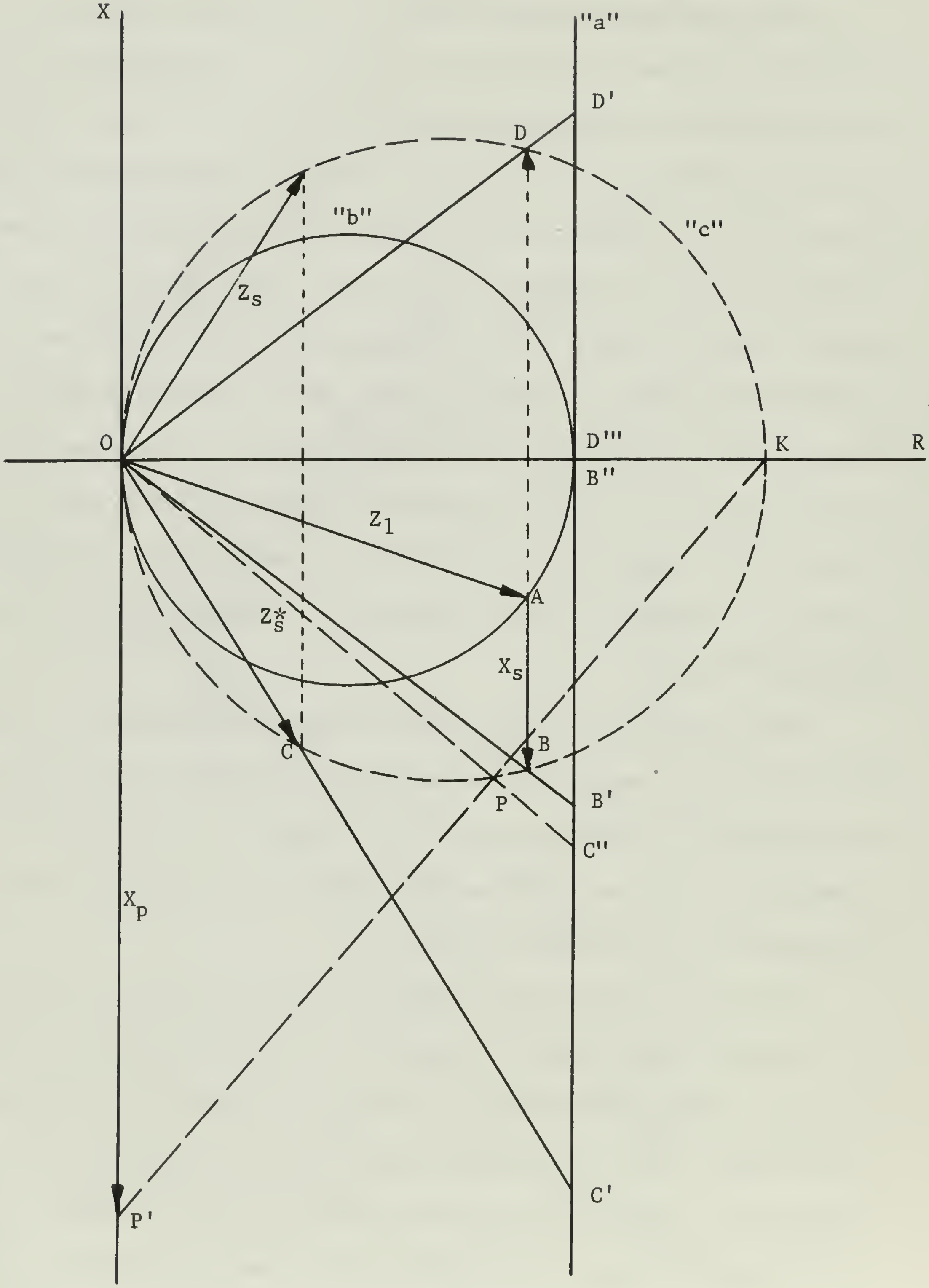
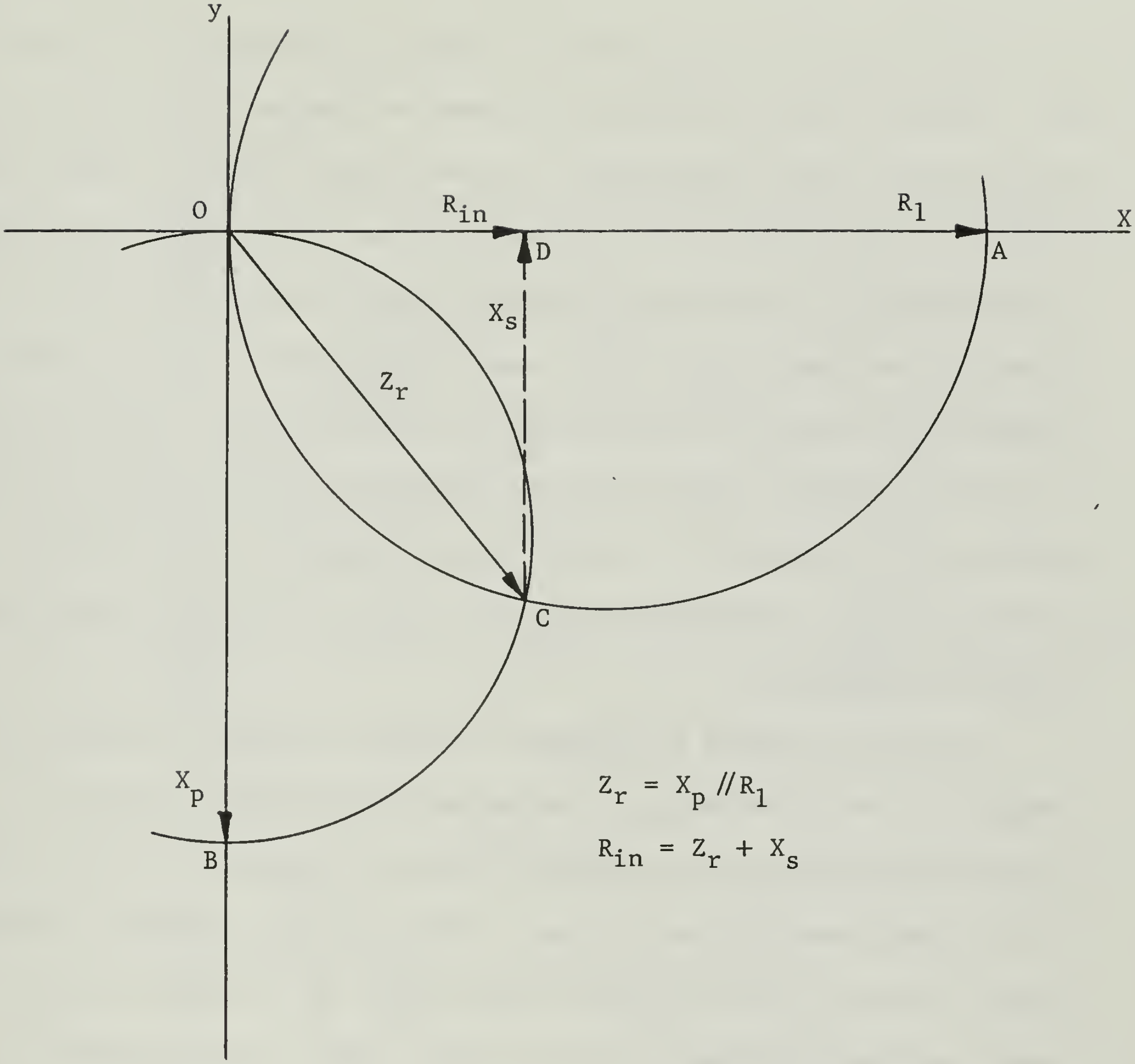


Figure 7-5. Example of Impedance Matching With "B"-Type L-Section



$Z_r = X_p // R_1$

$R_{in} = Z_r + X_s$

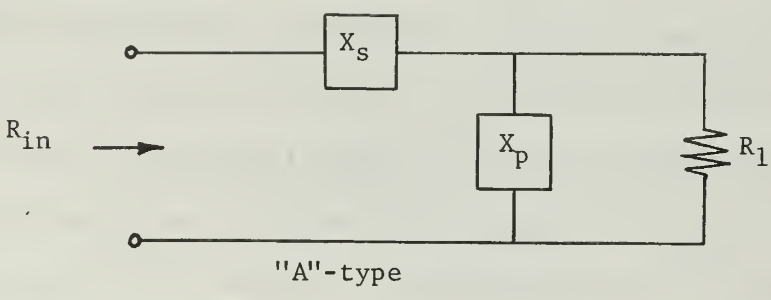


Figure 7-6. "A"-Type L-Network to Match Load R_1 to Source R_{in}

the x-axis tangent to the y-axis in O and going through A, the tip of R_1. Another circle is sketched that must be tangent to the x-axis at O and that must pass through point C on the original circle, where C has such a location that its projection on to the x-axis gives the desired resistance R_{in}. The second circle cuts the y-axis at point B and the phasor OB represents X_p, in this case a capacitance. Z_r is now the parallel equivalent of R_1 and X_p and it has the reactive component DC. X_s is the reactance that cancels DC, represented by the phasor CD, in this case an inductance. A second A-type L-section can be found by starting with an alternative point C' in the first quadrant and proceeding correspondingly.

Secondly a network is designed to match a resistive load R_1 to a complex source impedance Z_s. This method is depicted in Figure 7-7. It can be seen from the given parameters R_1 and Z_s that in this case either A or B-type L-sections can be used. Here an A-type network is chosen. First a circle is drawn tangent to the y-axis at O, centered on the x-axis and going through A, the tip of R_1. Then a pure reactance X_s of the value EF, here representing an inductance, is put in series with Z_s as to change it to Z_r. Z_r has now to be changed further, with the help of a shunt reactance X_p, so that Z_r in parallel with X_p will give R_1. This is done by drawing a circle tangent to OC at O and passing through A. It intersects with the y-axis at D and phasor OD represents X_p, in this case a pure inductance. A second set of values for X_s and X_p can be found by changing Z_s to OC' in the first step of the construction.and then proceeding correspondingly.

The general case where a complex load Z_1 is matched to a complex source impedance Z_s is described next. With load and source impedance

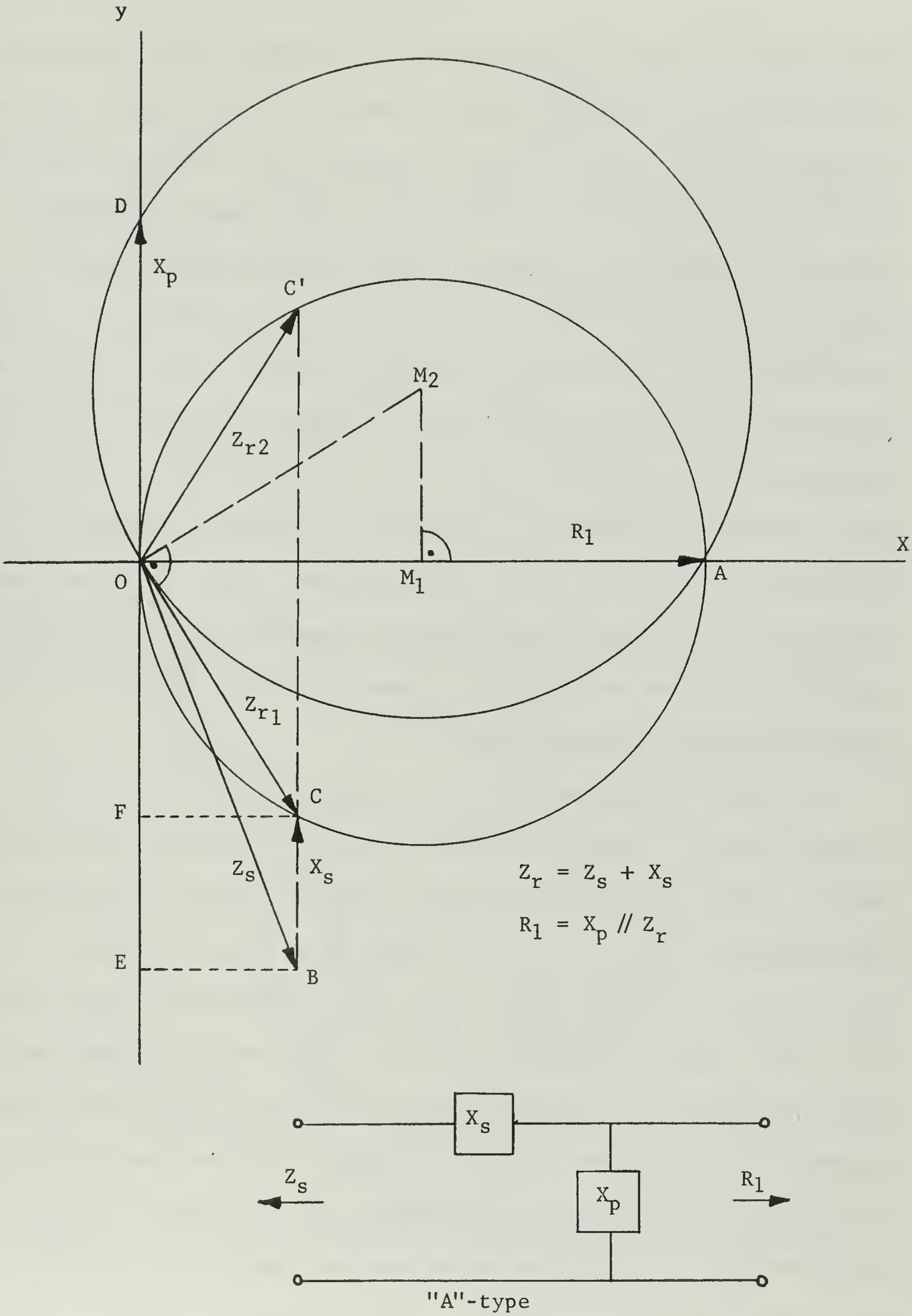


Figure 7-7. "A"-Type L-Network to Match Load R_1 to Complex Source Impedance Z_s

being complex it must be recalled that for maximum power transfer one wants the load Z_1 to see the source Z_s as Z_1^*, or vice versa the source Z_s to see the load Z_1 as Z_s^*. For the case depicted in Figure 7-8 either A- or B-type L-sections can be used, but an A-type network is chosen. First the circle tangent to the y-axis at O and passing through A and A' is sketched. Then a pure reactance, in this case an inductance, is put in series with Z_s as to get from B to C on the circle. Z_r is now the impedance which when paralleled with a still unknown reactance X_p will give as equivalent Z_1^*. To get X_p another circle is drawn tangent to OC at O and going through A'. The phasor OD is then the desired X_p, where D is the intersection of the just drawn circle with the y-axis. A second construction of an A-type network with different component values is possible. X_s has to be chosen so that Z_s is changed to OC' and the rest of the construction follows correspondingly.

In Figure 7-9 a construction is presented that has the job to find a network that will match a resistive load R_1 to a complex source impedance Z_s. The same problem was already solved in Figure 7-7 with an A-type L-section. Here it will be solved with a B-type L-section. R_1 and Z_s in Figure 7-9 are thus that only a B-type network will achieve matching. First a pure reactance X_p has to be found which when paralleled with Z_s will give Z_r, the phasor OC, with C lying on the circle that is tangent to the y-axis at O and goes through B, the real part of OC being the load resistance R_1. X_p, represented by OD, is found with help of a circle tangent to OB at O and going through C and its intersection with the y-axis fixes point D. Finally a pure reactance X_s has to be added in series to cancel the reactive component of Z_r to give R_1. X_s is represented by the phasor CA. Here, too, a second solution is possible in

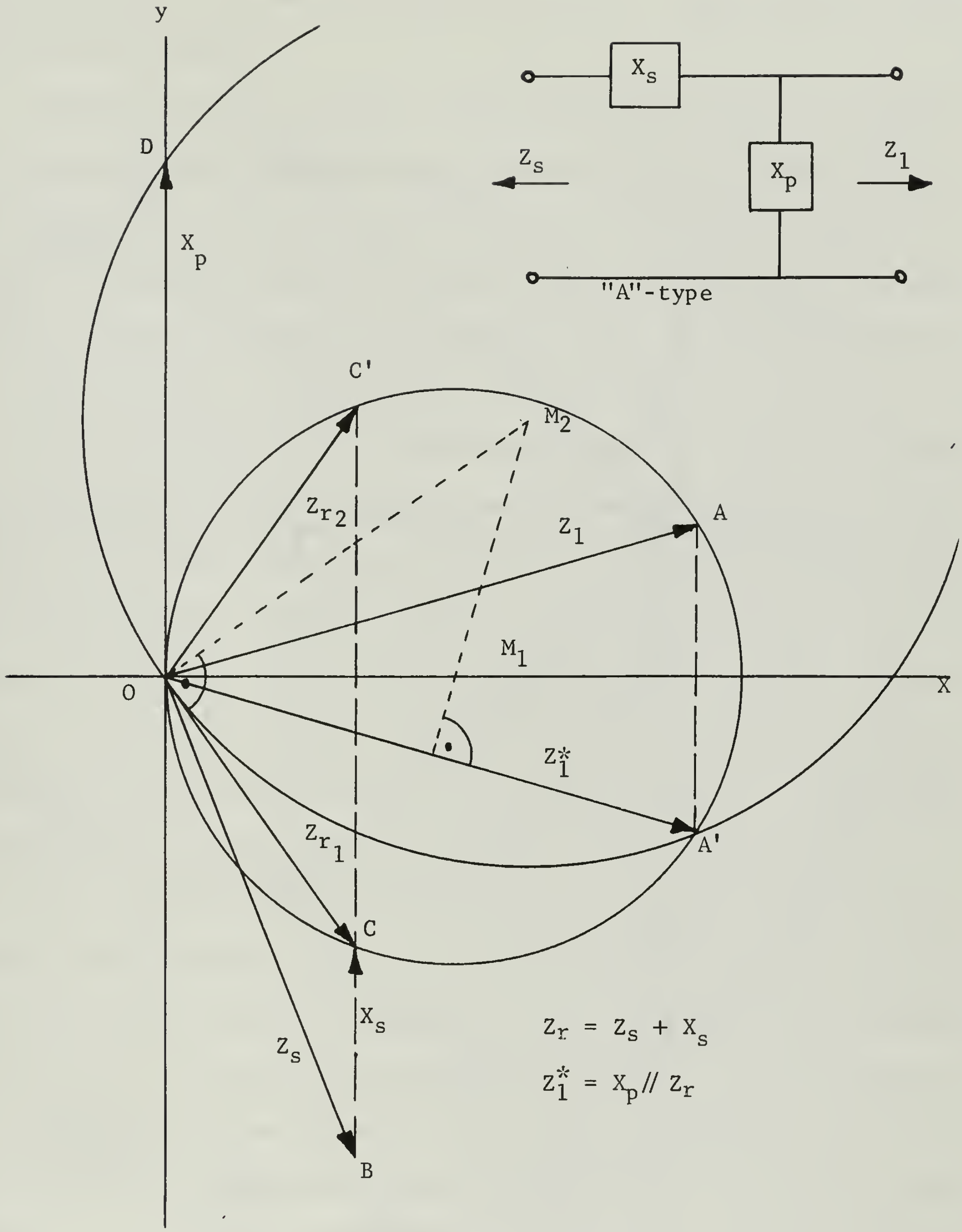
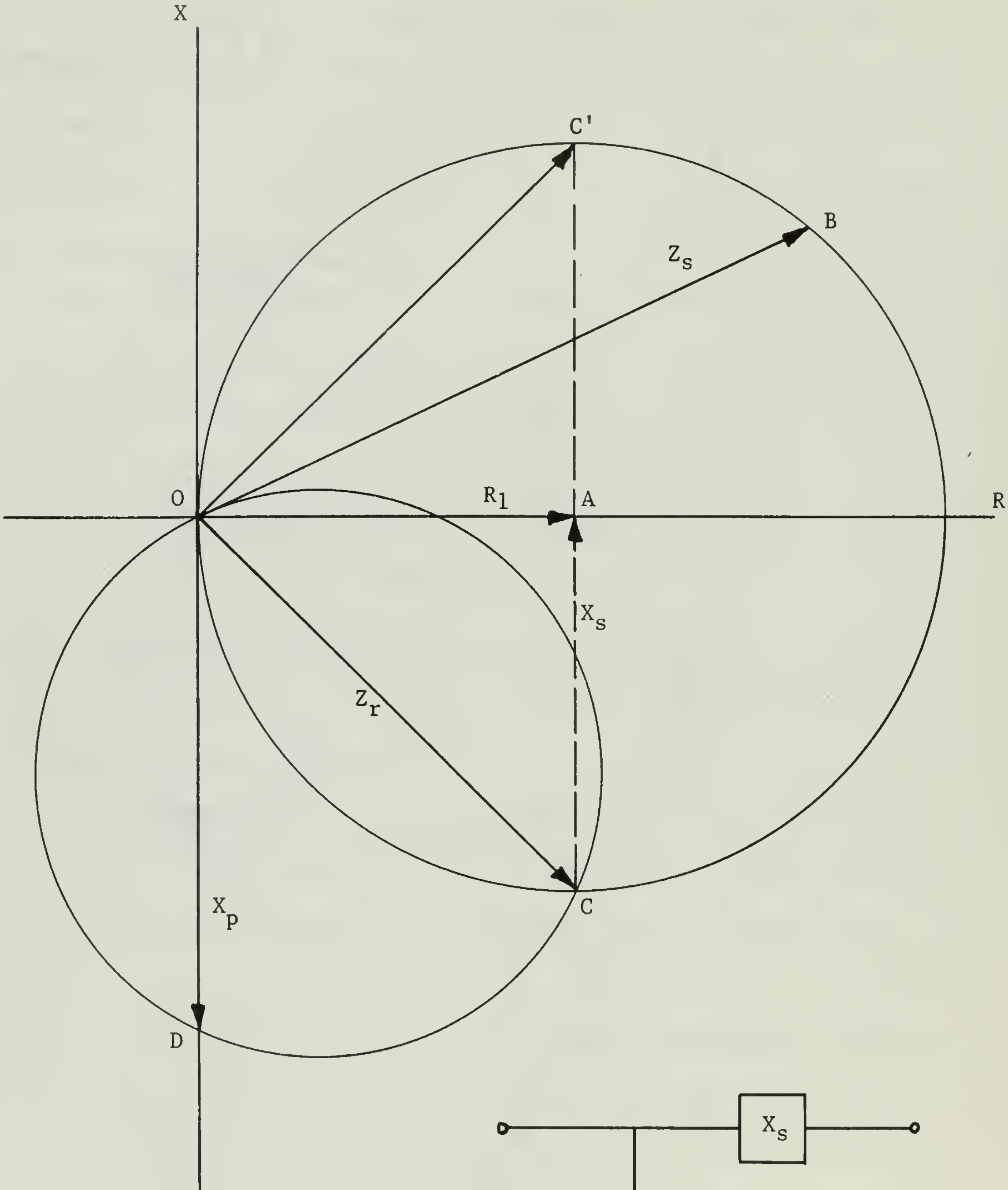
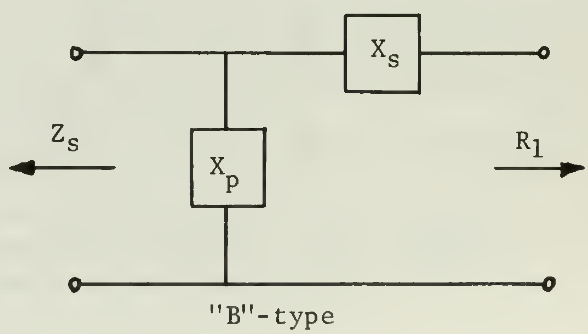


Figure 7-8. "A"-Type L-Network to Match Complex Load Impedance Z_1 to Complex Source Impedance Z_s





$Z_r = X_p // Z_s$

$R_1 = Z_r + X_s$

Figure 7-9. "B"-Type L-Network to Match R_1 to Complex Source Impedance Z_s

choosing point C' instead of C and preforming the remaining part of the construction similarly.

The general case in which a complex source impedance Z_s is matched to a complex load Z_l with a B-type L-section is shown in Figure 7-10. The construction follows along the same line as the one of Figure 7-9. However, it has to be noted that to get point C a perpendicular through E must be drawn, where OE is the real part of Z_l, and the series addition of X_s to Z_r has to yield Z_l^*, the conjugate of Z_l.

The third section of this chapter is devoted to a method introduced by P. J. Selgin. In this method the load Z_l is transformed to the conjugate of the source impedance Z_s with X_s and X_p, the two components of an L-section matching network. However a new principle is applied by Selgin. The Z-point is moved from Z_l to Z_s^* using the fact that the addition of a pure reactive series arm must leave R_l, the real part of Z_l, constant, and that the addition of a pure reactive shunt arm must leave G unchanged. In moving from Z_l to Z_s^* the Z-point follows constant R lines and constant G lines only. Depending on the values of Z_l and Z_s either or both L-section types can be used to realize the matching network and for each type of L-section there exist two different solutions. In Figure 7-11 a B-type matching section is required. To transform Z_l into Z_s^* one starts out at point A, moves along the constant R_l line to B and then along the constant G circle clockwise to C' via C and L. The desired series value Z_{sl} is represented by the phasor AB. The desired parallel value X_{pl} is a little bit more difficult to obtain, because it must be kept in mind that the constant G circle was used to get from B to C. The values of the reactances at points B and C must be known. This can be achieved by drawing the constant B-circles, both having their

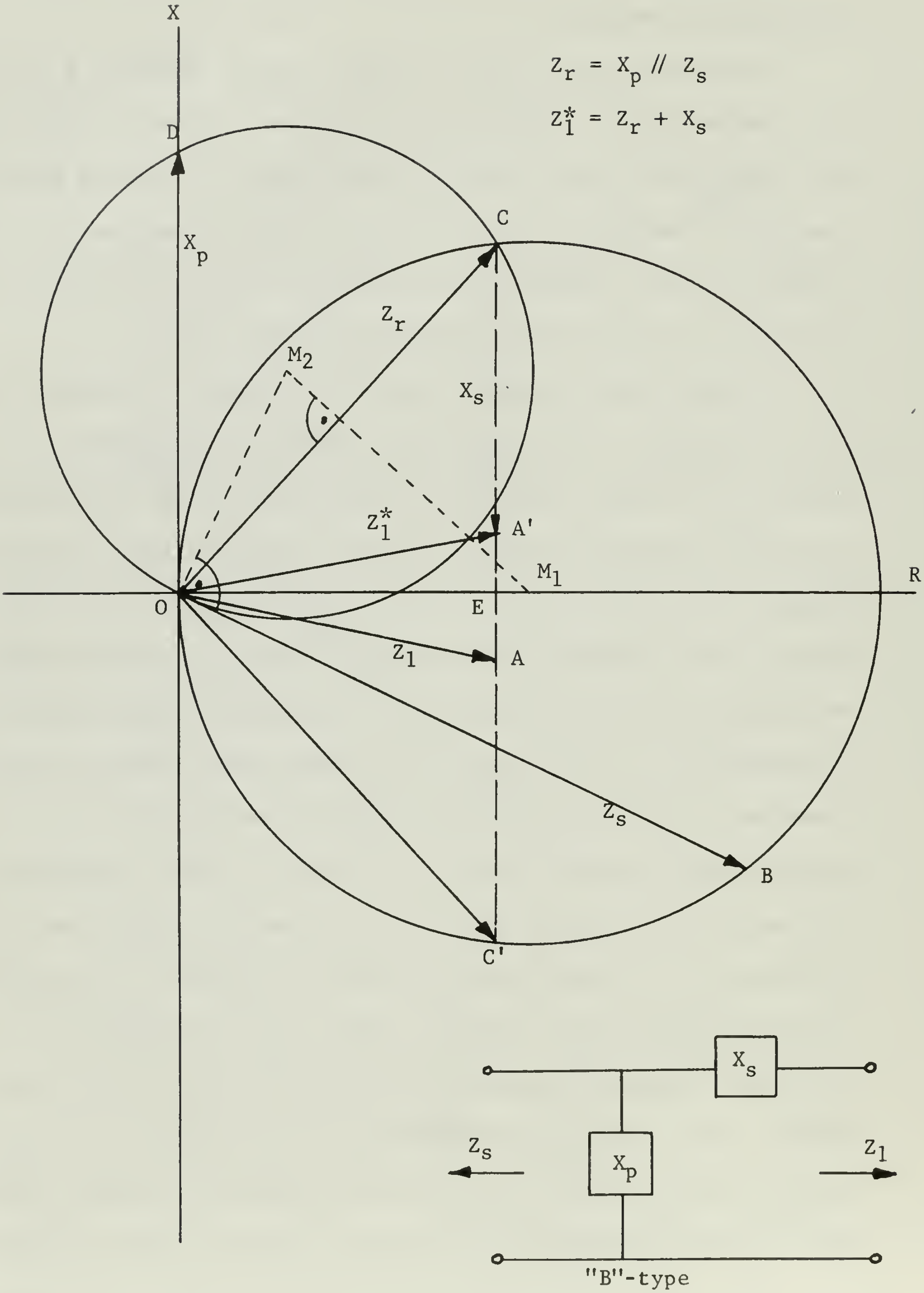


Figure 7-10. "B"-Type L-Network to Match Complex Load Impedance Z_1 to Complex Source Impedance Z_s

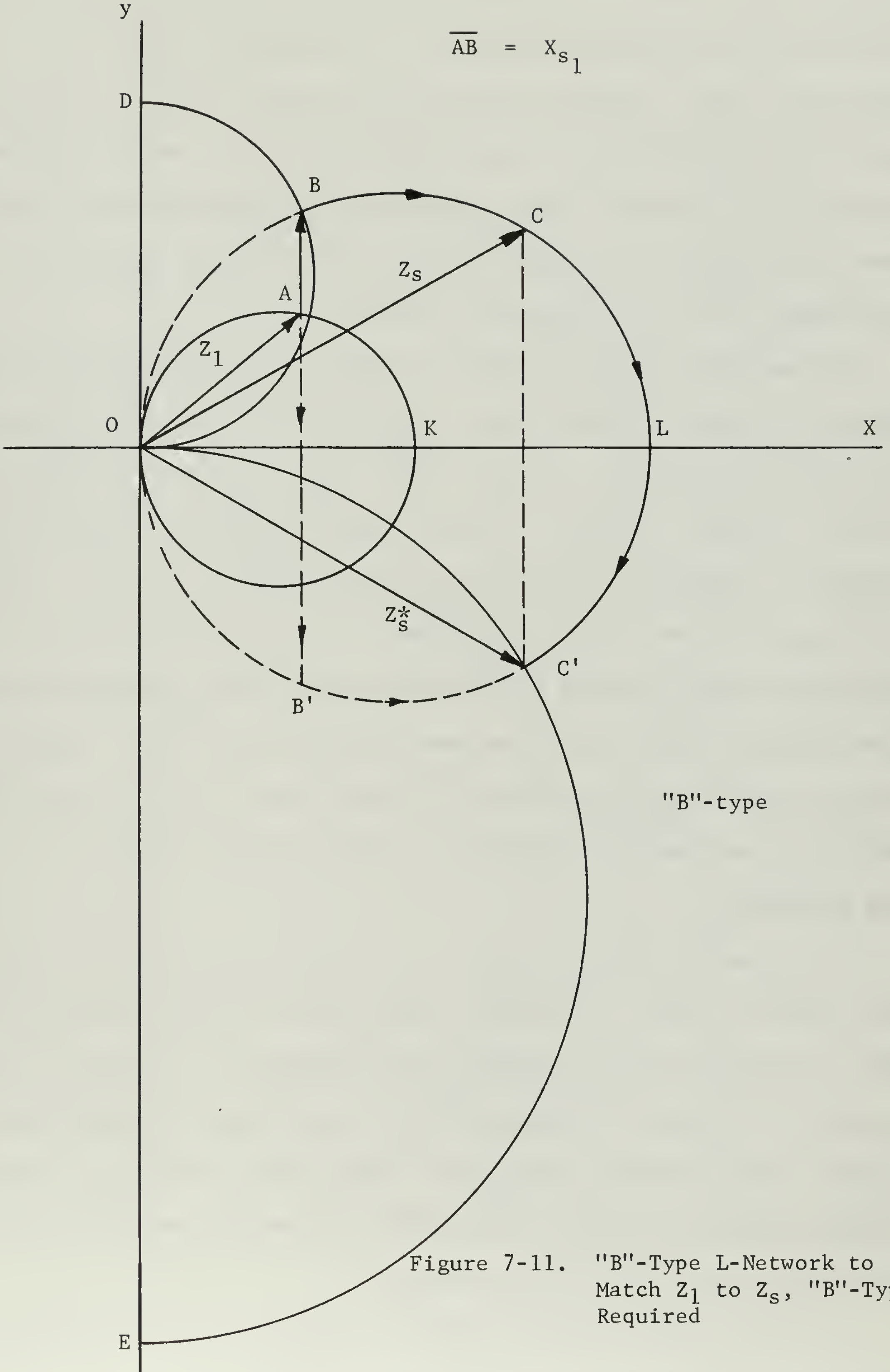


Figure 7-11. "B"-Type L-Network to Match Z_1 to Z_s, "B"-Type Required

centers on the y-zxis, one passing through point B and cutting the y-axis at D, the other going through C' and crossing the y-axis at E. The value $B_{p1} = 1/OD - (-1/OE)$ is positive in this case and $X_{p1} = -1/B_{p1}$ is negative which means that a shunt capacitor has to be used to move the Z-point to C'. A second B-type L-section with different values can be found in a similar fashion, by going from A to B' on the R_1-line and then from B' to C' on the constant G'circle.

The matching network in Figure 7-12 can only be implemented with an A-type L-section. The shunt value X_{p1} must first be brought into parallel with Z_1. To get B_{p1} one has to move from A to B on the constant G-circle. Then $B_{p1} = (-1/OE) - (-1/OD)$, a positive value because in this example OE is greater than OD. The desired value $X_{p1} = -1/B_{p1}$, a negative value which means that a capacitance has to be taken. From point B the Z-point must be shifted to C' along the constant R_s-line and the desired X_{s1} that can do this is represented by the phasor BC'. Another A-type L-section can be constructed by going from A via K to B' and thus getting X_{p2} and then by following the constant R_s-line from B' to C' thus getting X_{s2}.

In the case depicted in Figure 7-13 Z_1 and Z_s are of such a nature that either an A-type or a B-type L-section can be taken as matching networks. A B-type section is chosen and the procedure to get X_{s1}, X_{s2} and X_{p1}, X_{p2} is the same as in Figure 7-11. X_{s1} is given by phasor AB and X_{p1} is a negative value, calling for a capacitance because $B_{p1} = 1/OD - (-1/OP)$ is positive. X_{s2} is represented by phasor AB' and will be a capacitance and X_{p2} will be an inductance.

In Figure 7-14 an A-type L-section is chosen for the same matching problem and the same procedure as in Figure 7-12 is followed. X_{s1} and

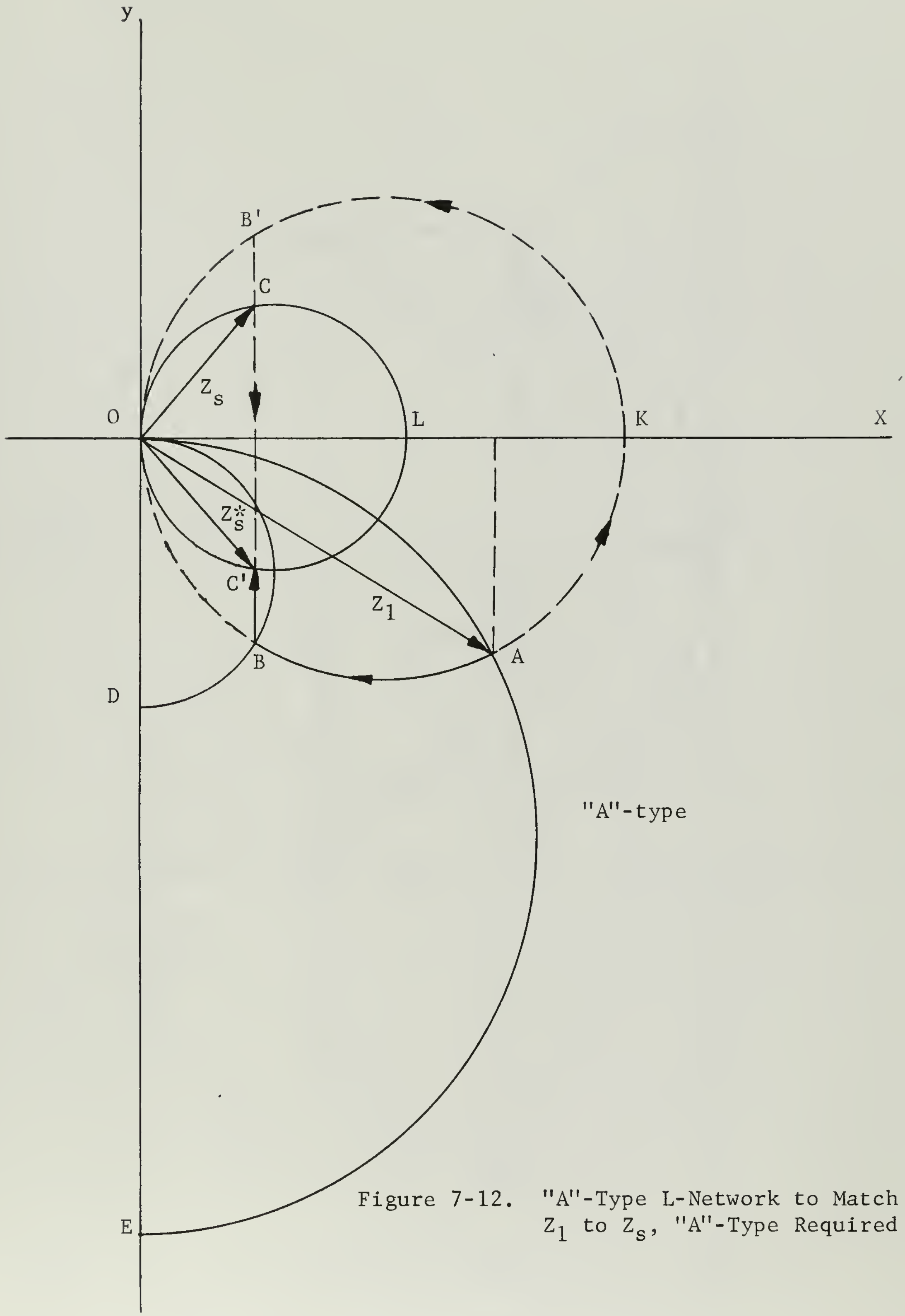
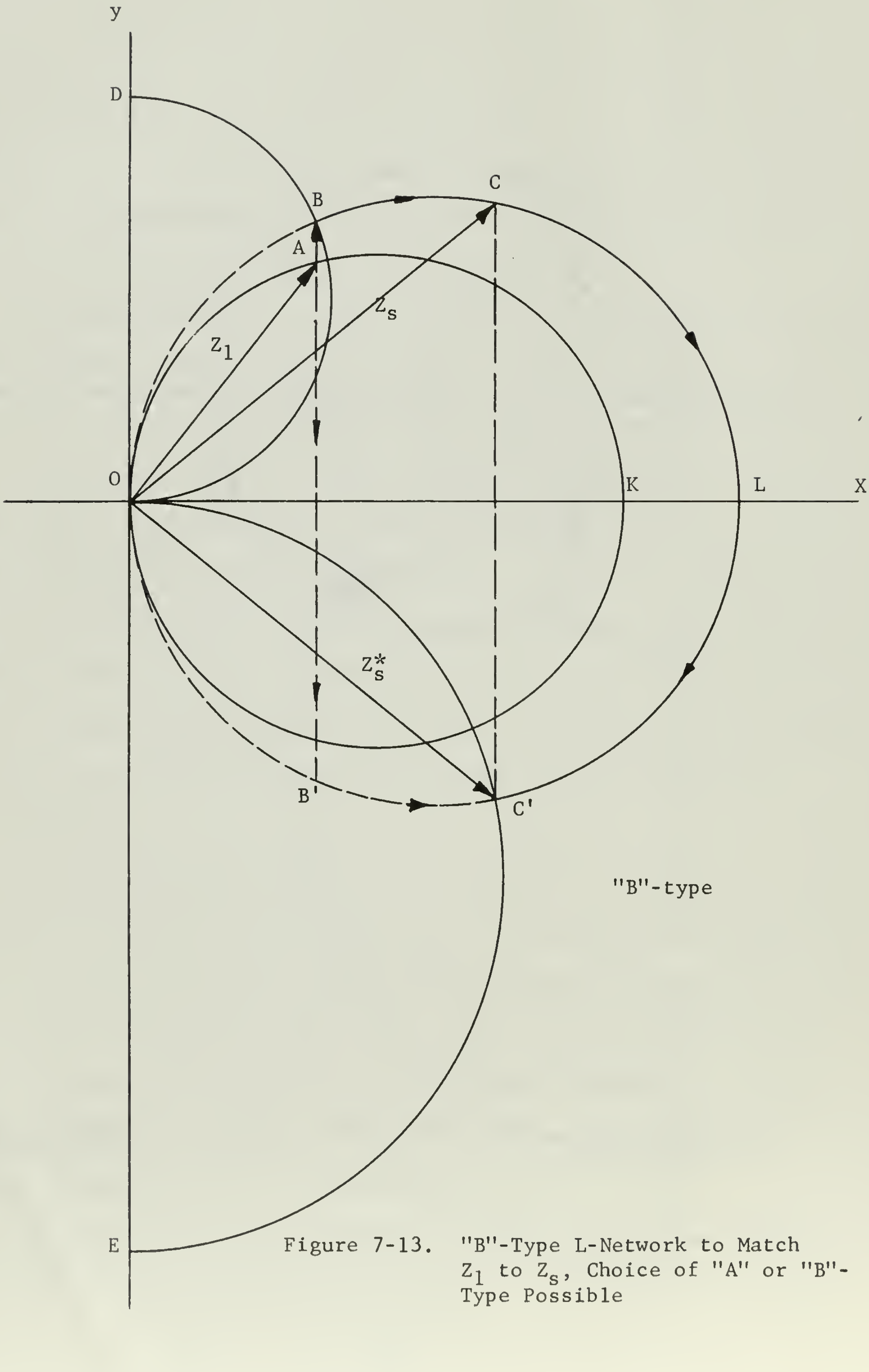


Figure 7-12. "A"-Type L-Network to Match Z_1 to Z_s, "A"-Type Required



"B"-type

Figure 7-13. "B"-Type L-Network to Match Z_1 to Z_s, Choice of "A" or "B"-Type Possible

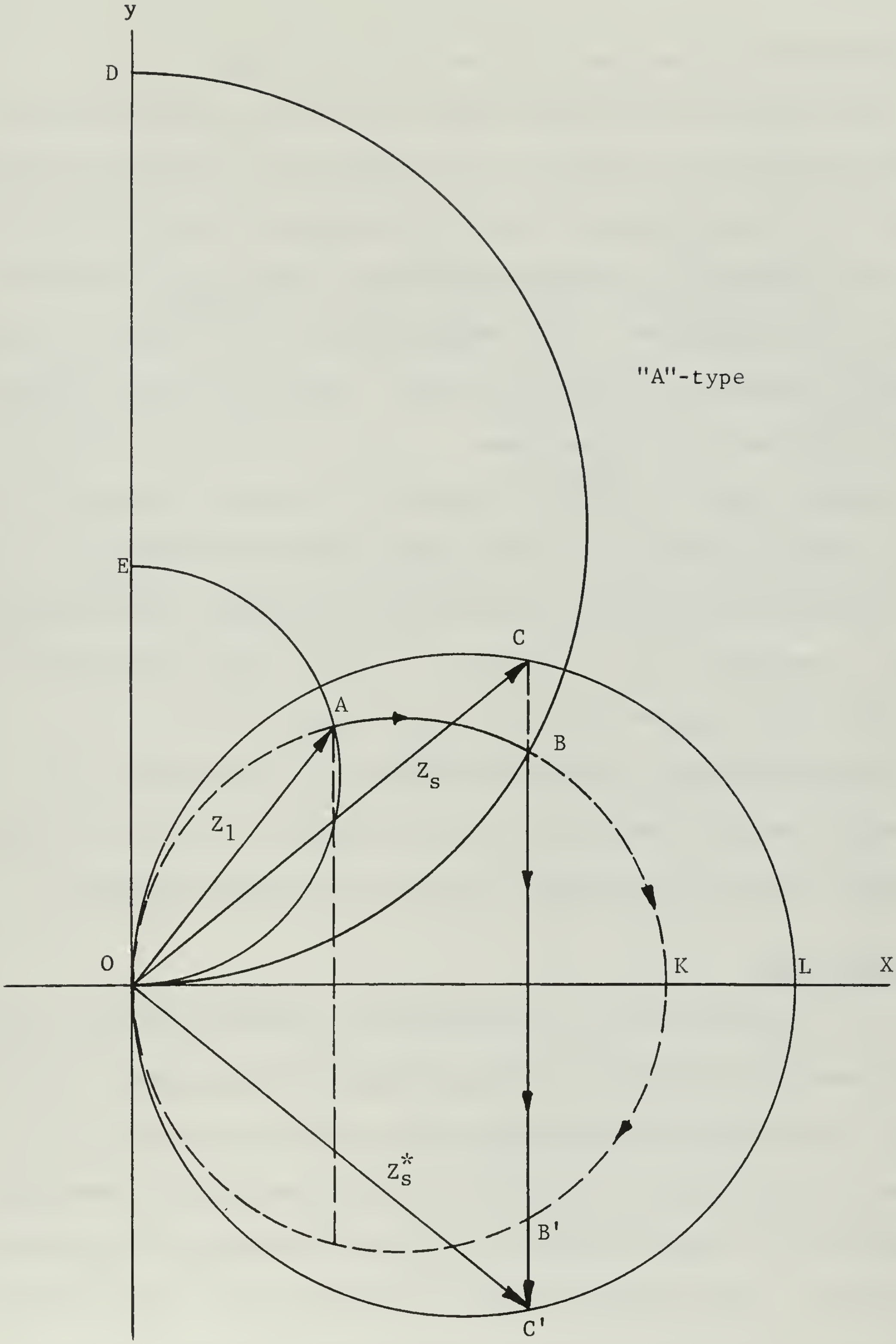


Figure 7-14. "A"-Type L-Network to Match Z_1 to Z_s, Choice of "A" or "B"-Type Possible

X_{p1} both will be capacitances and so will X_{s2} and X_{p2}.

The fourth section brings a method that is not comparable to the methods worked out up to here. In this method the impedances are represented by voltage and current vectors and because of the synthesis of T- and Pi-sections as matching networks more flexibility is achieved than with L-sections only. The method that will be presented in form of numerical examples originates with Laport. Before proceeding further some conventions will be laid down:

-- Different scales are used for currents and voltages; however, all currents have to be drawn to the same scale, so have all voltages.

-- Voltage and current vectors representing the load are taken as references. From there on the problem will be worked towards the input of the section by using vector addition and/or subtraction.

-- The counterclockwise direction is taken to be the positive one.

-- Current through a capacitor leads the voltage by 90 degrees; voltage through an inductance leads the respective current by 90 degrees.

Resistive Load.

1. A 500 Ohm load is to be matched to a 100 Ohm resistance with a -30 degrees phase shift between input and load current using a T-section. Figure 7-15 shows the block diagram, Figure 7-17 the solution. I_1 is chosen to be 1.0 ampere and V_1 to be 500 volts resulting in convenient scales and in a load of 500 ohms. The power dissipated in the load W_1 is 500 watts. Since there is to be a -30 degrees phase shift through the network, next the input voltage and current vectors are drawn in phase, leading the load by 30 degrees. The ratio of input voltage to input current should represent the input impedance of 100 ohms. To find the exact length of these two vectors it must be remembered that when

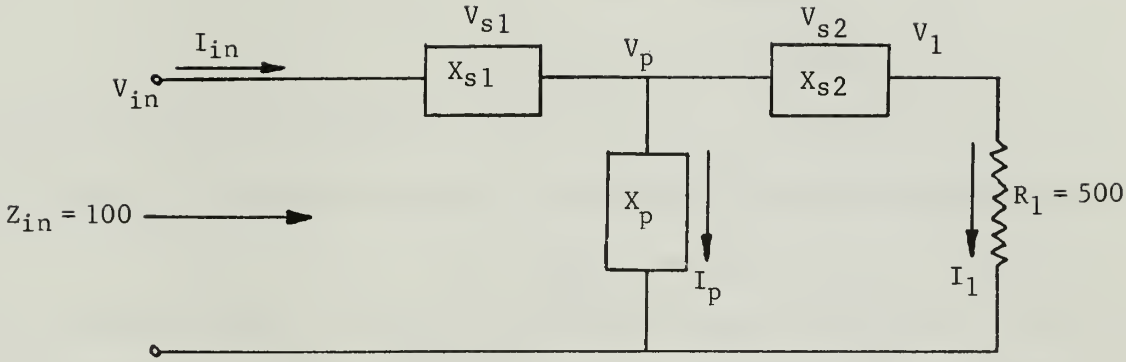


Figure 7-15. T-Network to Match Resistive Load R_1 to Resistive Source Z_{in}

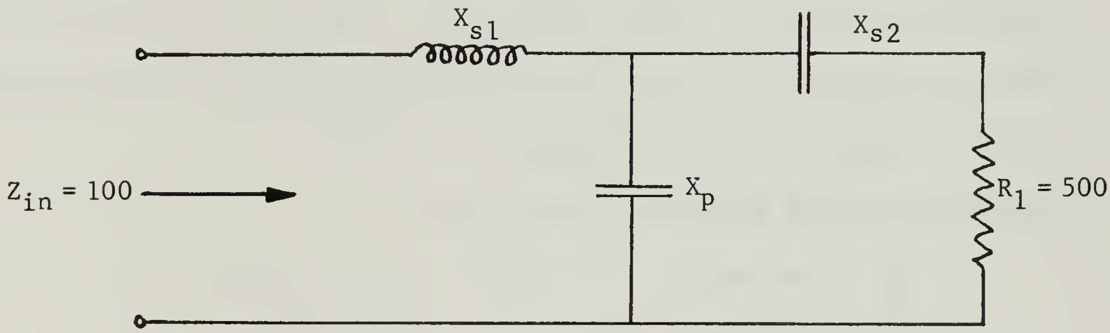


Figure 7-16. Final Matching Circuit to Figure 7-15

pure reactances are taken in the matching network, the output power must be equal to the input power. Since $P_i = 500$ watts and $Z_i = 100$ ohms it follows $P_i = V_i I_i$, $P_i Z_i = V_i Z_i I_i$, $P_i Z_i = V_i^2$,

$V_i = \sqrt{P_i Z_i} = \sqrt{500 \times 100} = 224$ volts and

$I_i = P_i / V_i = 500 / 224 = 2.24$ amperes.

Now V_1, R_1, I_1, V_i, I_i, Z_i are known. By using Kirchhoff's voltage and current laws one can add vectorally looking at Figures 7-15 and 7-17

$V_1 + V_{s2} = V_p$, $V_p + V_{s1} = V_i$, $I_1 + I_p = I_i$.

Next a connection is drawn between the tips of I_1 and I_i leading to I_p immediately. The proper direction follows from Kirchhoff's current equations. Since voltages and currents through reactances are 90 degrees out of phase, perpendiculars are drawn to I_1 through the tip of V_1 and to I_i through the tip of V_i. Their intersection fixes V_p. Completing the vector triangles corresponds to fulfilling Kirchhoff's voltage equation so that a solution has been obtained.

Resulting directly from the diagram are:

$V_1 = 500$ volts	$I_1 = 1.0$ amp
$V_{s2} = 412$ volts	$I_p = 1.49$ amp
$V_p = 665$ volts	$I_i = 2.24$ amp
$V_{s1} = 624$ volts	
$V_i = 224$ volts	

Now: $X_{s1} = V_{s1}/I_i = j624 / 2.24 = j279$ ohms , an inductance.

$X_{s2} = V_{s2}/I_1 = -j412 / 1.0 = -j412$ ohms

$X_p = V_p / I_p = -j665 / 1.49 = -j446$ ohms , both capacitances.

The complete, synthesized T-section matching network is shown in Figure 7-16.

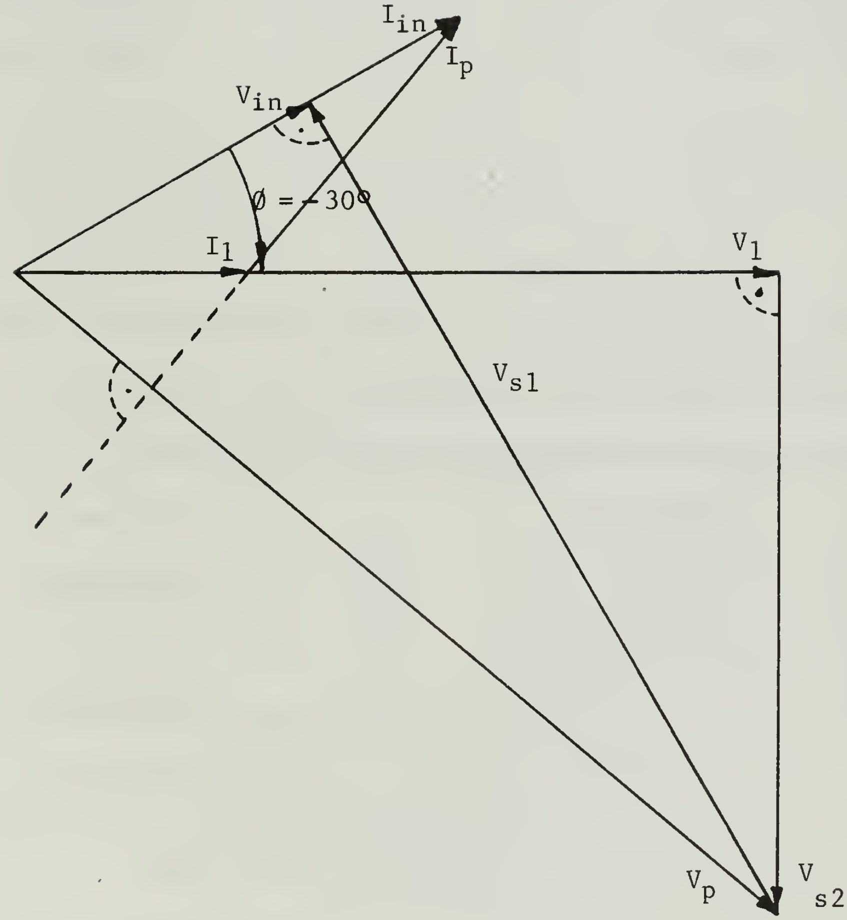


Figure 7-17. Actual Construction to Figure 7-15.

2. A 500 ohm load is to be matched to a 100 ohm resistance with a -30 degrees phase shift between input and load current using a Pi-section. The block diagram with all voltages and currents is given in Figure 7-18, the actual solution is shown in Figure 7-20. The construction is along the same line as the one for the T-section. As before V_i, I_i and V_1, I_1 are known. First the voltage triangle is formed according to Kirchhoff's voltage equations: $V_i = V_1 + V_s$. Since I_{p2} leads V_1 by 90 degrees and I_{p1} leads V_i by 90 degrees two perpendiculars are erected, to V_1 at the tip of I_1, to V_i at the tip of I_i. Their intersection fixes I_s.

$$I_s = I_1 + I_{p2}$$

$$I_i = I_s + I_p$$

The direction of the currents follows from Kirchhoff's current equations. The result can be checked by examining if V_s leads I_s by 90 degrees. From the sketch the following results can be taken:

V_1 = 500 volts	I_1 = 1.0 amp
	I_{p2} = 2.73 amp
V_s = 331 volts	I_s = 2.9 amp
	I_{p1} = 1.83 amp
V_i = 224 volts	I_1 = 2.24 amp

Then:

$X_{p1} = V_i/I_{p1} = 224/-j1.83 = j122.5$ ohms

$X_s = V_s/I_s = j331/2.9 = j114$ ohms, both inductances and

$X_{p2} = V_1/I_{p2} = 500/j2.73 = -j183$ ohms, a capacitance.

The complete, synthesized Pi-section matching network is shown in Figure 7-19.

3. A 500 ohm load is to be matched to a 100 ohm resistance with no

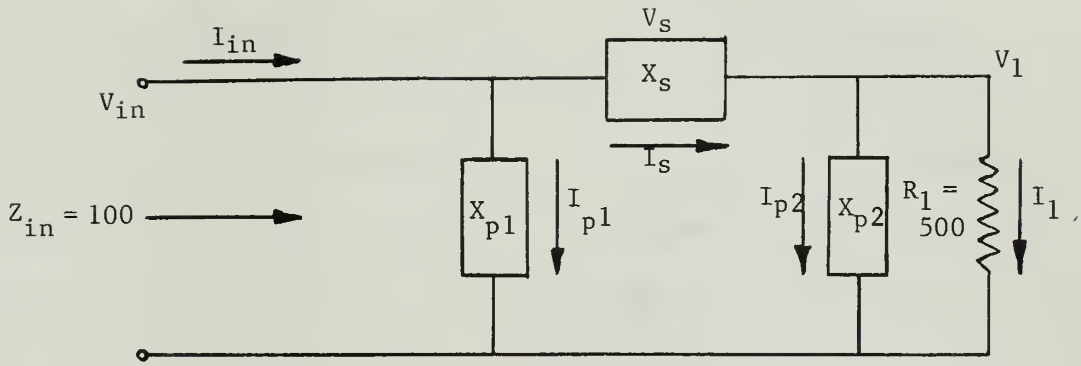


Figure 7-18. Pi-Network to Match Resistive Load R_1 to Resistive Source Z_{in}

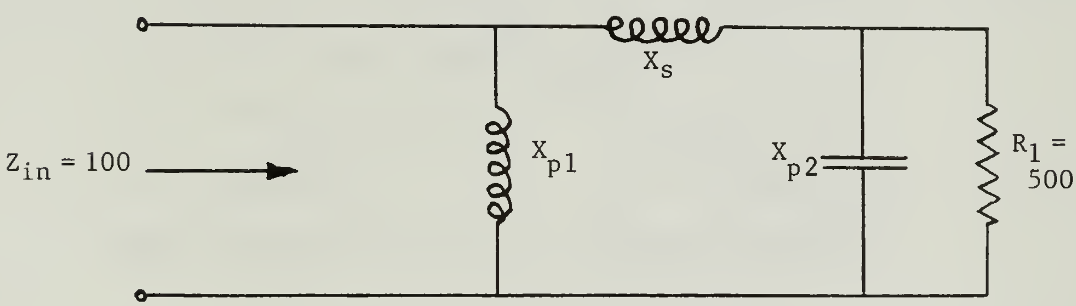


Figure 7-19. Final Matching Network to Figure 7-18

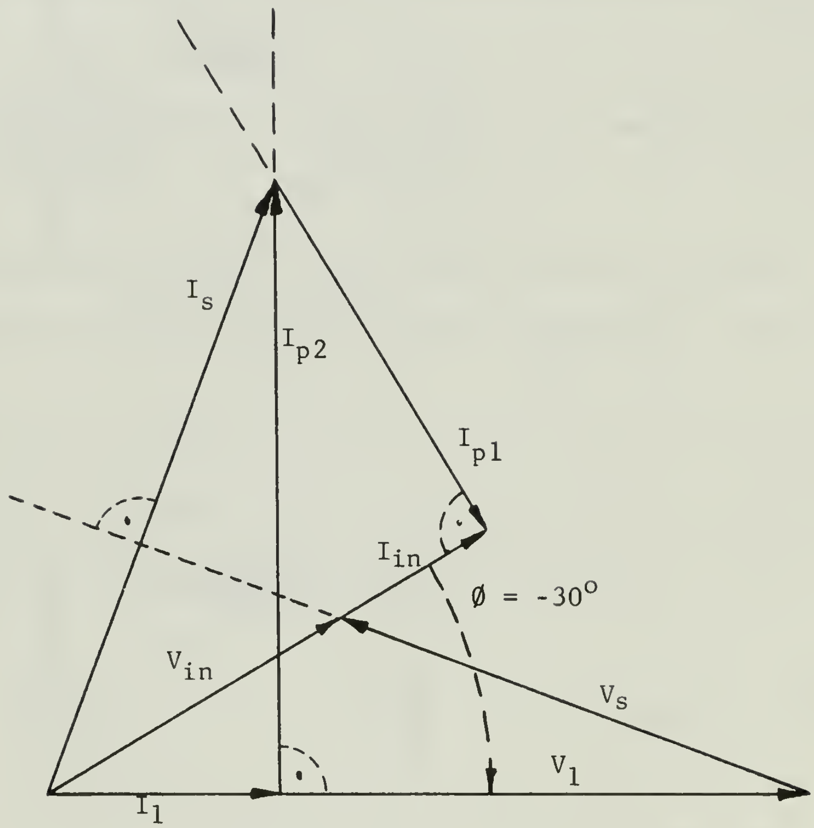


Figure 7-20. Actual Construction to Figure 7-18

phase shift specification between input and output current using an L-section.

Figure 7-20 indicates that I_{p1} could be made zero, which corresponds to X_{p1} being equal to infinity and Figure 7-17 indicates that V_{s2} could be made equal to zero so that X_{s2} would become zero. Therefore I_s in the first diagram would become I_i and V_p in the latter diagram would become V_1. Then in Figure 7-20 V_s will be perpendicular to I_i and in Figure 7-17 I_p will be perpendicular to V_1. So the two diagrams merge into a single one depicted in Figure 7-21. Its construction consists of drawing V_1 and I_1 as before, of sketching circles with radius V_i and I_i respectively around the origin. Then a perpendicular to V_1 at the end of I_1 is erected and the origin is connected with its intersection with the I_i-circle. V_1 is connected to the intersection of I_i with the V_i-circle. This last operation yields V_s which is perpendicular to I_i. Kirchhoff's voltage and current laws are fulfilled. From the diagram the following results can be taken:

V_1 = 500 volts	I_1 = 1.0 amp
V_s = 448 volts	I_p = 2.0 amp
V_i = 224 volts	I_i = 2.24 amp

Then:

$X_s = V_s/I_i = j448/2.24 = j200$ ohms, an inductive reactance and

$X_p = V_1/I_p = 500/j2.0 = -j250$ ohms, a capacitive reactance.

The complete L-section is depicted in Figure 7-22.

<u>Complex Impedance Load</u>.

1. A complex load $Z_1 = 75 - j30$ is to be matched to an impedance $Z_s = 600 - j150$ introducing a $+60^\circ$ phase-shift between input and load current.

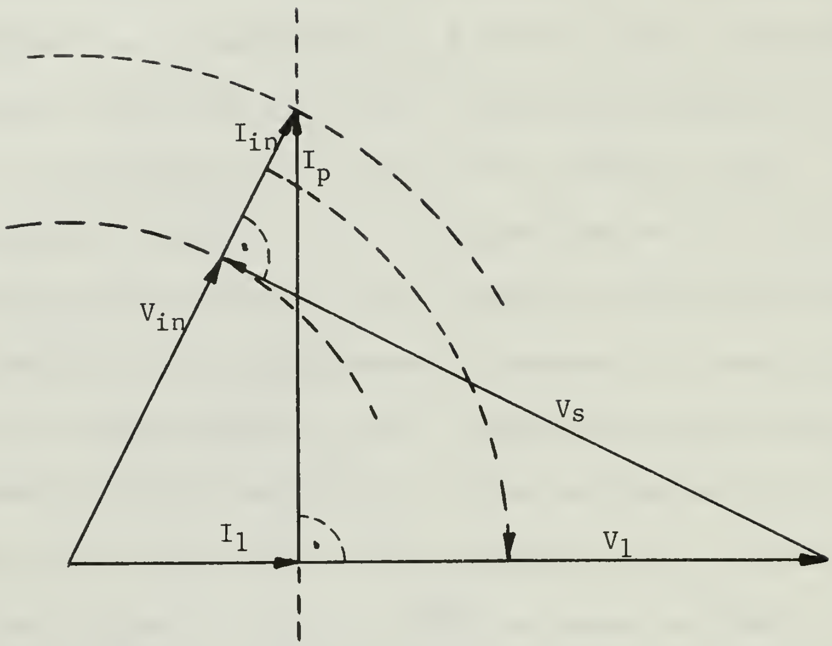


Figure 7-21. L-Section to Match Resistive Load R_1 to Resistive Source Z_{in}, Actual Construction

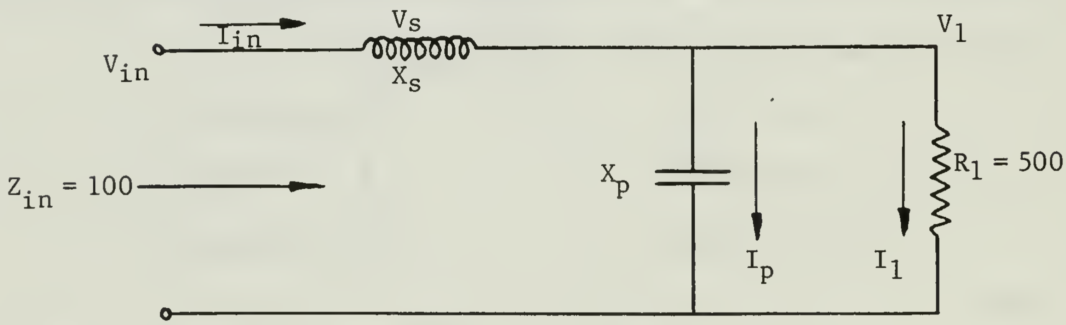


Figure 7-22. Final Matching Circuit to Figure 7-21

A matching problem of this type is handled in such a way that series or parallel reactances are added to the load to make it appear as the complex conjugate of the source impedance. Convenient scales are chosen:

$I_1 = 1.0$ amp

$P_1 = V_{1r} I_1 = 75$ watts

$V_{ir} = \sqrt{P_i R_i} = \sqrt{75 \times 60} = 212$ volts

$V_{ix} = I_i X_i = 0.354 \times j150 = j53$ volts

$V_i = \sqrt{(212)^2 + (53)^2} = 219$ volts

$V_{1r} = 75$ volts

$V_{1x} = -j30$ volts

$P_i = 75$ watts

$I_i = \sqrt{P_i / R_i} = \sqrt{75/600} = 0.354$ amp

First the T-section is chosen hereby referring to Figure 7-23 for the block-diagram and to Figure 7-25 for the actual construction using Kirchhoff's voltage and current equations.

From the diagram the following values result:

$V_1 = 81.0$ volts	$I_1 = 1.0$ amp
$V_i = 219$ volts	$I_p = 0.875$ amp
$V_{s1} = 91$ volts	$I_1 = 0.354$ amp
$V_{s2} = 173$ volts	
$V_p = 215$ volts	

Then:

$X_{s1} = V_{s1}/I_i = j91/0.354 = j264$ ohms

$X_p = V_p/I_p = j215/0.875 = j246$ ohms, both inductances,

$X_{s2} = V_{s2}/I_1 = 173/j1.0 = -j173$ ohms, a capacitance.

The complete matching network is shown in Figure 7-24.

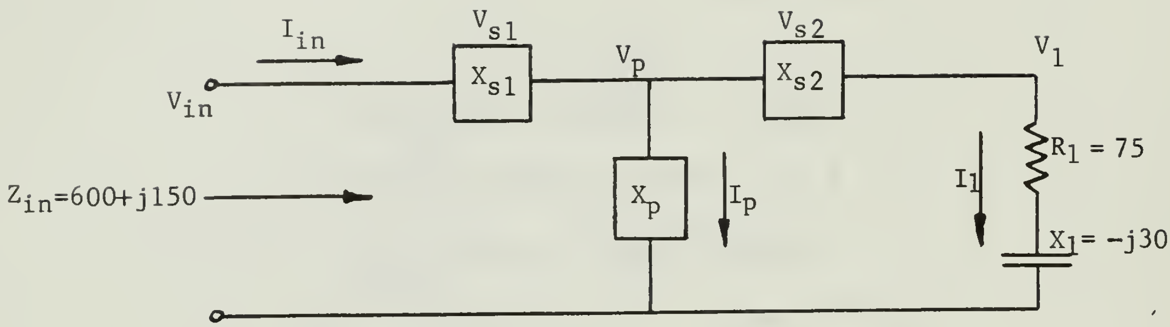


Figure 7-23. T-Network to Match Complex Load Z_1 to Complex Source Z_{in}

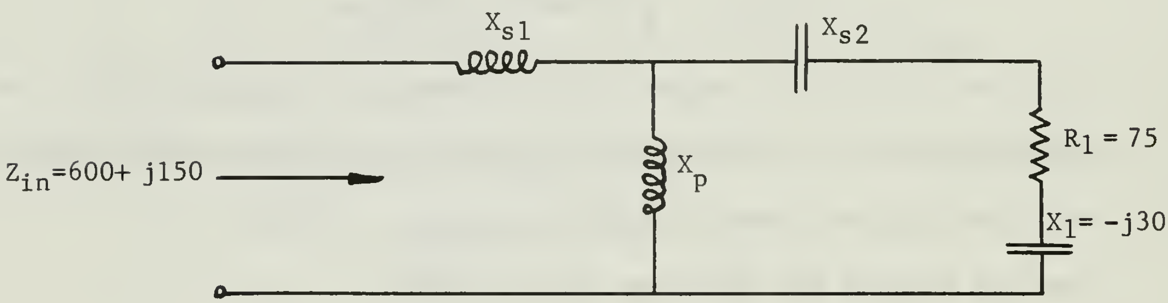


Figure 7-24. Final Matching Circuit to Figure 7-23

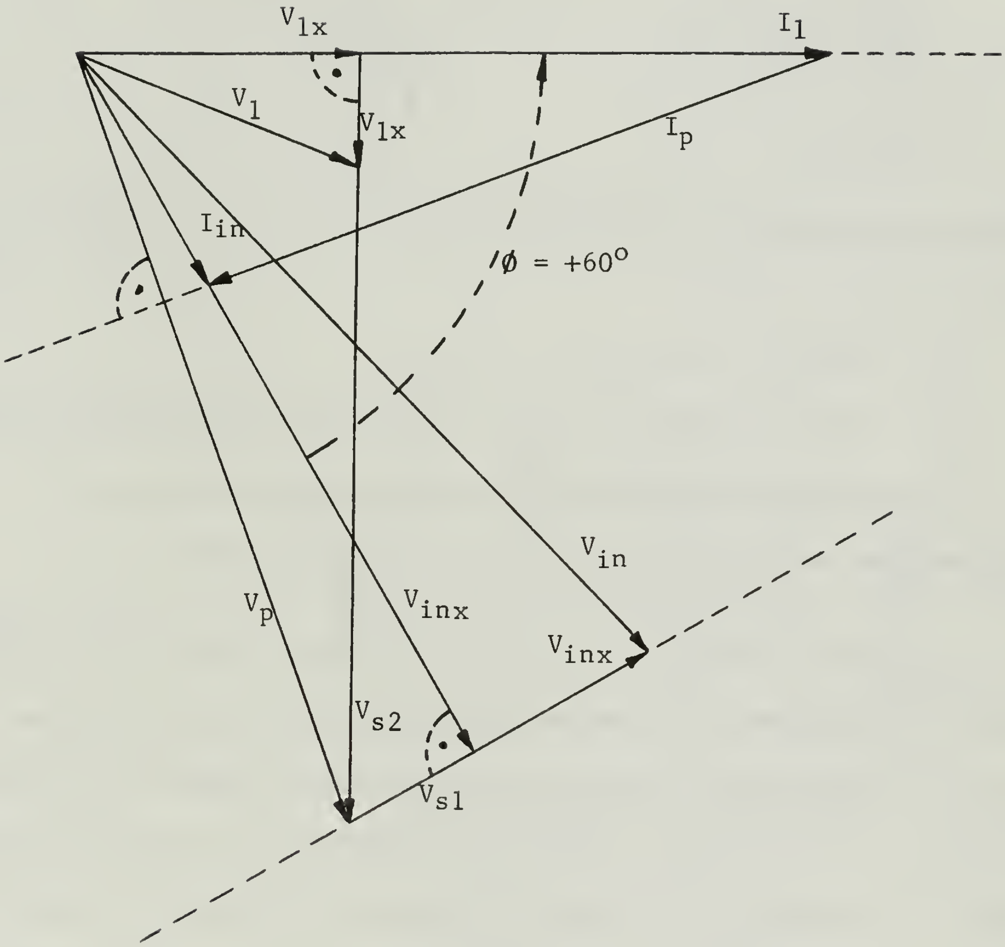


Figure 7-25. Actual Construction to Figure 7-23

2. Next a Pi-network is used as shown in Figure 7-26. The construction of voltage and current triangles following Kirchhoff's rules is depicted in Figure 7-28. Directly from the diagram follows:

V_1 = 81 volts	I_s = 1.57 amp
V_i = 219 volts	I_{p1} = 1.62 amp
V_s = 147 volts	I_{p2} = 0.89 amp
	I_1 = 1.0 amp
	I_i = 0.354 amp

Given these voltages and currents the matching network reactances are calculated:

$X_{p1} = V_i/I_{p1} = j219/1.62 = j135$ ohms

$X_s = V_s/I_s = j147/1.57 = j03.5$ ohms, both representing inductances,

$X_{p2} = V_1/I_{p2} = 81/j0.89 = -j91$ ohms, a capacitance.

The complete matching network is given in Figure 7-27.

3. If this problem is to be solved with no phase shift specification using an L-section it can be seen from Figure 7-3 that a B-type section has to be used which results in the same block-diagram as in Figure 7-23 with $X_{s1} = 0$. The other solution, corresponding to the block-diagram in Figure 7-23 with $X_{s2} = 0$ cannot be used since Kirchhoff's current law cannot be fulfilled with I_p perpendicular to V_1 as would be necessary. Referring to the block-diagram in Figure 7-29 and the actual solution in Figure 7-30 the following values result:

V_1 = 81 volts	I_1 = 1.0 amp
V_s = 174 volts	I_p = 1.03 amp
V_i = 219 volts	I_i = 0.354 amp

These give reactance values of:

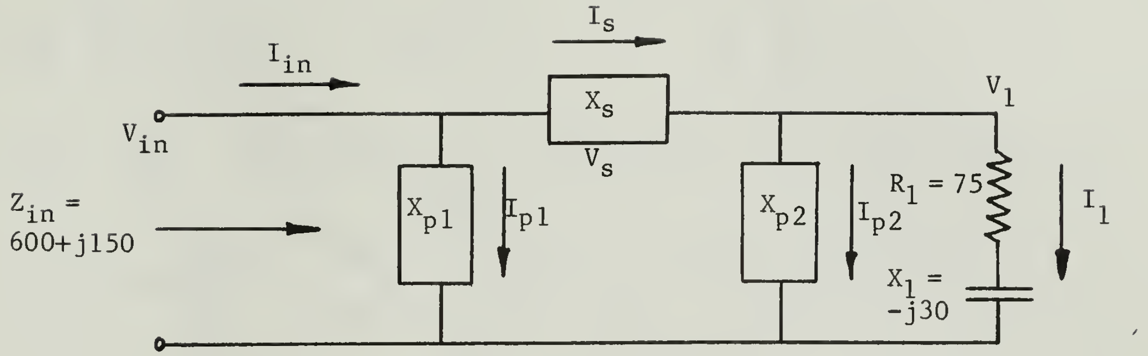


Figure 7-26. Pi-Network to Match Complex Load Z_1 to Complex Source Z_{in}

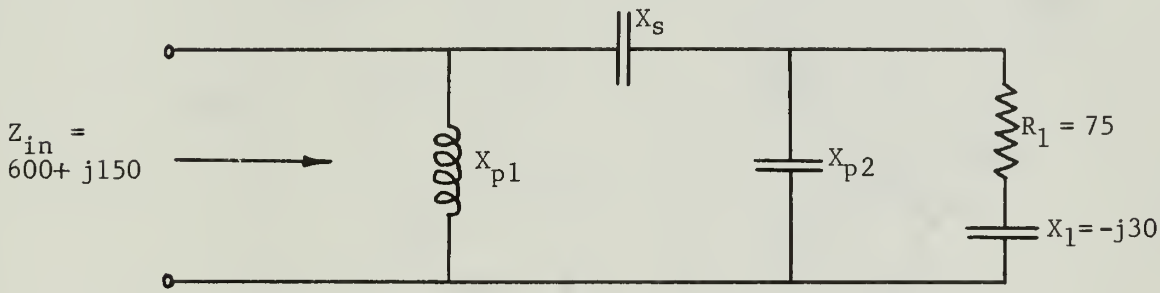


Figure 7-27. Final Matching Circuit to Figure 7-26

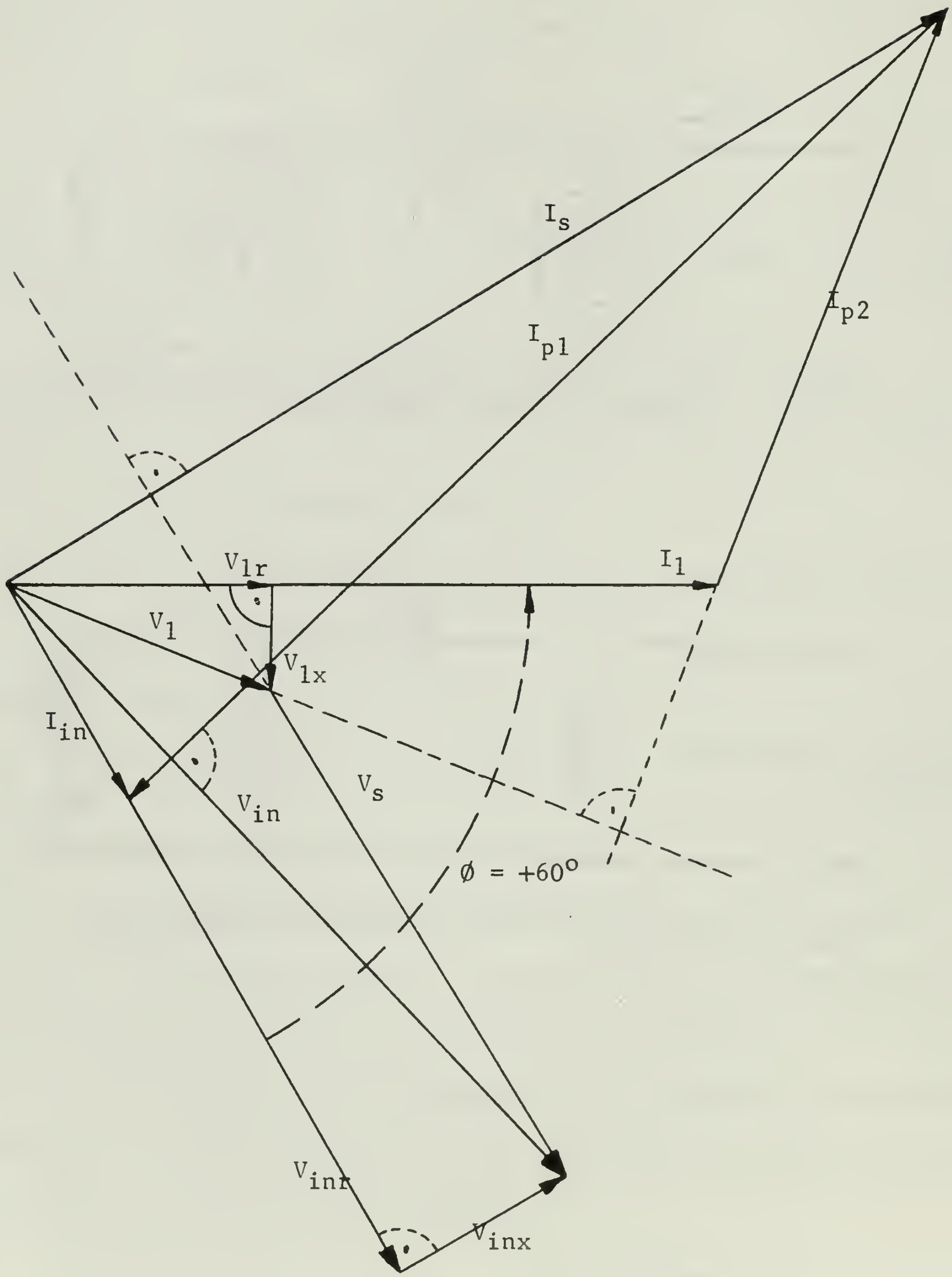


Figure 7-28. Actual Construction to Figure 7-26

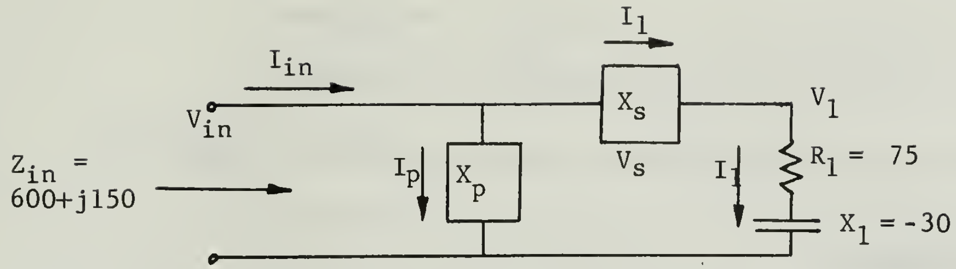


Figure 7-29. L-Network to Match Complex Load Z_1 to Complex Source Z_{in}

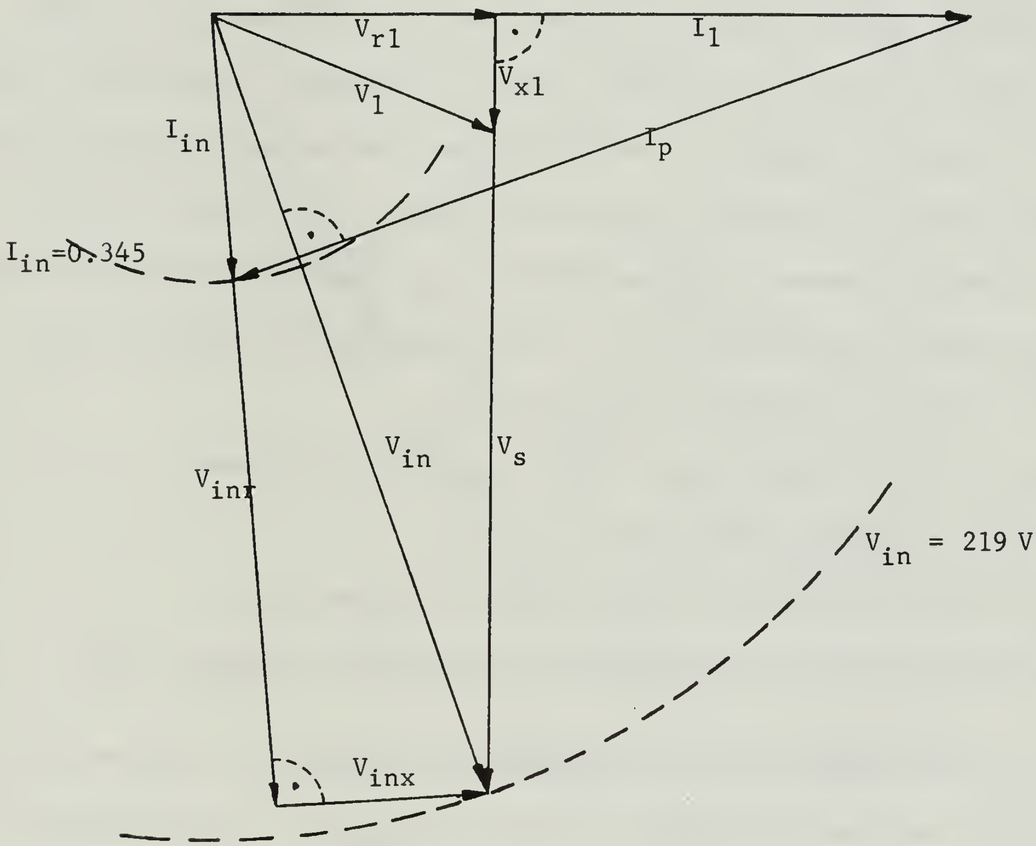


Figure 7-30. Actual Construction to Figure 7-29

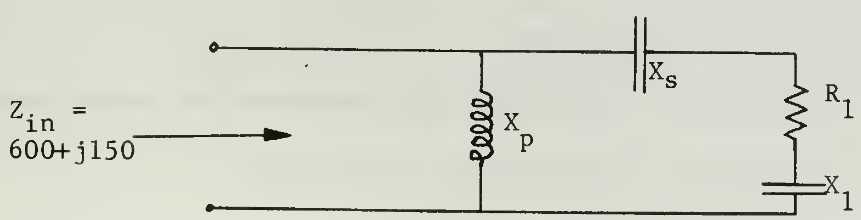


Figure 7-31. Final Matching Circuit to Figure 7-29

$X_s = V_s/I_1 = 174/j1.0 = -j174$ ohms, a capacitance,

$X_p = V_i/I_p = j219/1.03 = j211$ ohms, an inductance.

The complete, synthesized matching network is shown in Figure 7-31.

Phase Shift Network.

1. A circuit is to be designed which will produce a -120^o phase shift between input and output current when $Z_1 = 100$ ohms and $Z_i = 100$ ohms. First a T-section is used to solve the problem. Referring to Figures 7-32 and 7-33, again Kirchhoff's voltage and current laws are applied. I_1 and V_1 must be in phase, so must be V_i and I_i. $V_1 = 100$ volts and $I_1 = 1.0$ amp are chosen for convenience. V_{s2} is made perpendicular to I_1 and V_{s2} is made perpendicular to I_1, their intersection results in V_p which in turn is perpendicular to I_p. Now:

$X_{s1} = V_{s1}/I_i = j173$ ohms $\qquad X_{s2} = V_{s2}/I_1 = j173$ ohms ,

both representing inductances,

$X_p = V_p/I_p = 200/j1.73 = -j116$ ohms, a capacitance.

The matching network in its final form is depicted in Figure 7-34.

2. Now a Pi-section is wanted and it is referred to Figures 7-35 and 7-36. Here perpendiculars I_{p2} on V_1 and I_{p1} on V_i are erected to yield:

$X_{p1} = V_i/I_{p1} = 100/j1.73 = -j58$ ohms

$X_{p2} = V_1/I_{p2} = 100/j1.73 = -j58$ ohms, both representing capacitances,

$X_s = V_s/I_s = 173/j2.0 = -j86.5$ ohms, an inductance.

The matching network is shown in Figure 7-37.

3. The method outlined above can be extended to the case where a complex instead of a resistive load is used. Phase shift can be

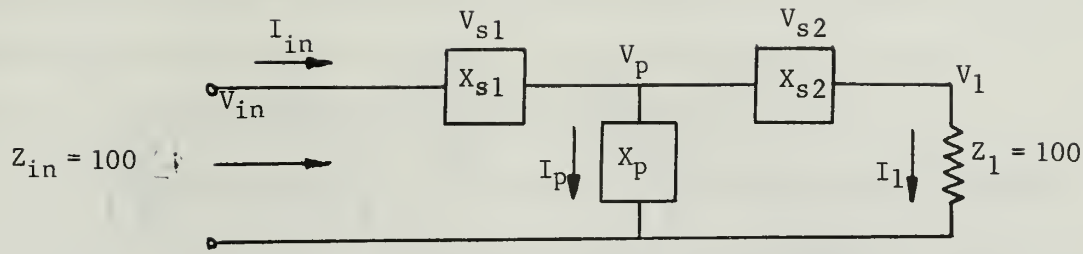


Figure 7-32. T-Network to Produce Desired Phase Shift Between Z_1 and Z_{in}

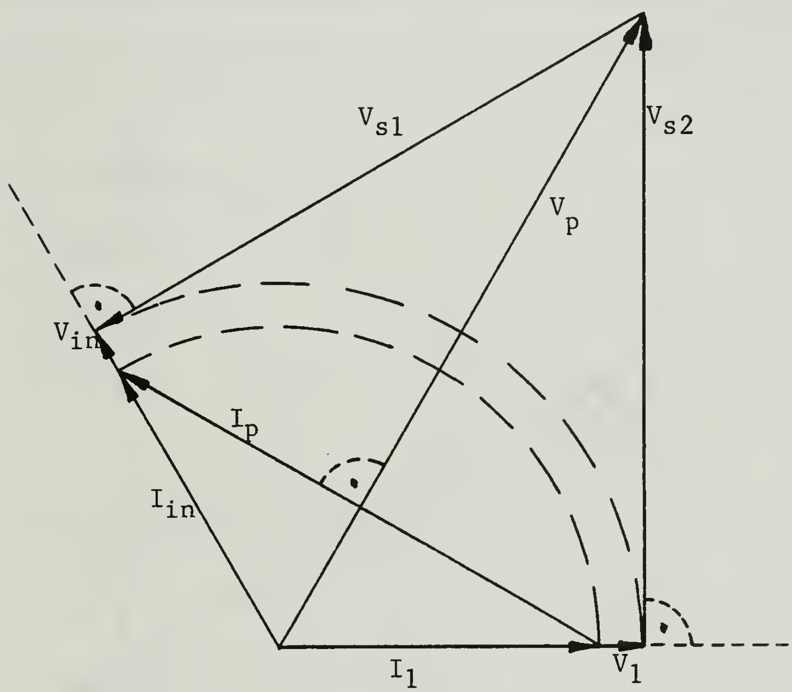


Figure 7-33. Actual Construction to Figure 7-32

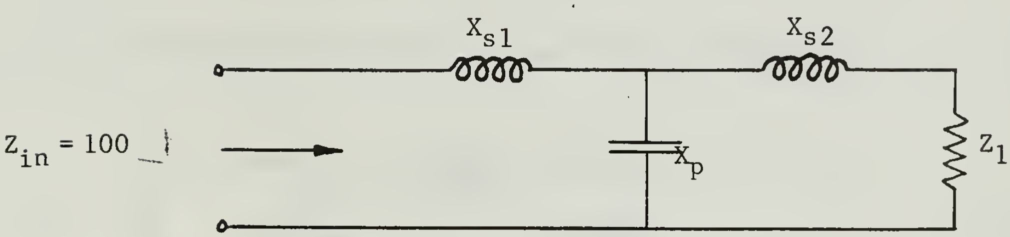


Figure 7-34. Final Phase Shift Circuit to Figure 7-32

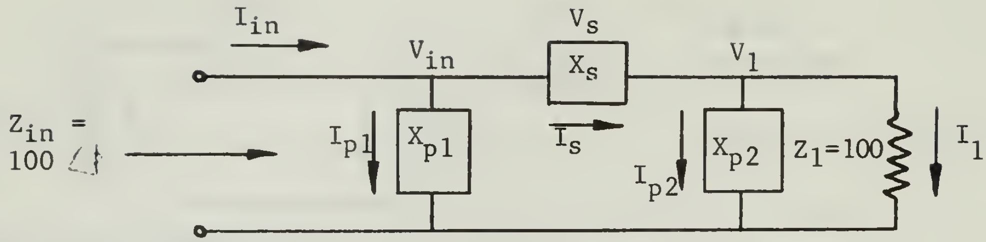


Figure 7-35. Pi-Network to Produce Desired Phase Shift Between Z_1 and Z_{in}

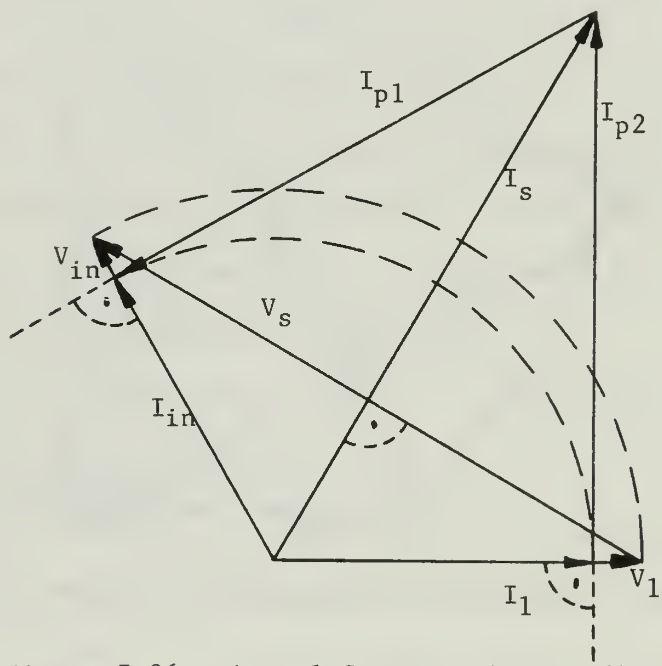


Figure 7-36. Actual Construction to Figure 7-35

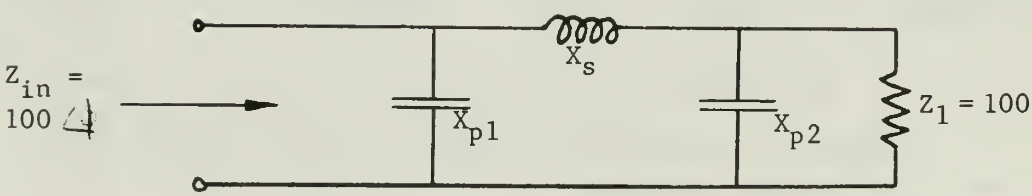


Figure 7-37. Final Phase Shift Circuit to Figure 7-35

introduced through the network, where the phase shift between input and load voltage and between input and load current must be the same, as otherwise the input impedance would be changed.

As an example $Z_1 = 75 - j30$ ohms and a network is to be designed introducing -120° phase shift between load and input voltage and between load and input current. For convenience a T-section will be used, though a Pi-network would be equally appropriate. The block-diagram has already been shown in Figure 7-23, the actual construction is according to Figure 7-38. From the diagram follows:

$I_p = 1.74$ amp	$V_1 = 81$ volts
$I_i = 1.0$ amp	$V_p = 150$ volts
$I_1 = 1.0$ amp	$V_{s1} = 100$ volts
	$V_{s2} = 160$ volts

These values give the following reactance values:

$X_p = V_p/I_p = 150/j1.74 = -j86.5$ ohms, a capacitance,

$X_{s1} = V_{s1}/I_i = j100/1.0 = j100$ ohms

$X_{s2} = V_{s2}/I_1 = j160/1.0 = j160$ ohms, both inductances.

The synthesized network is of the same form as the one in Figure 7-34.

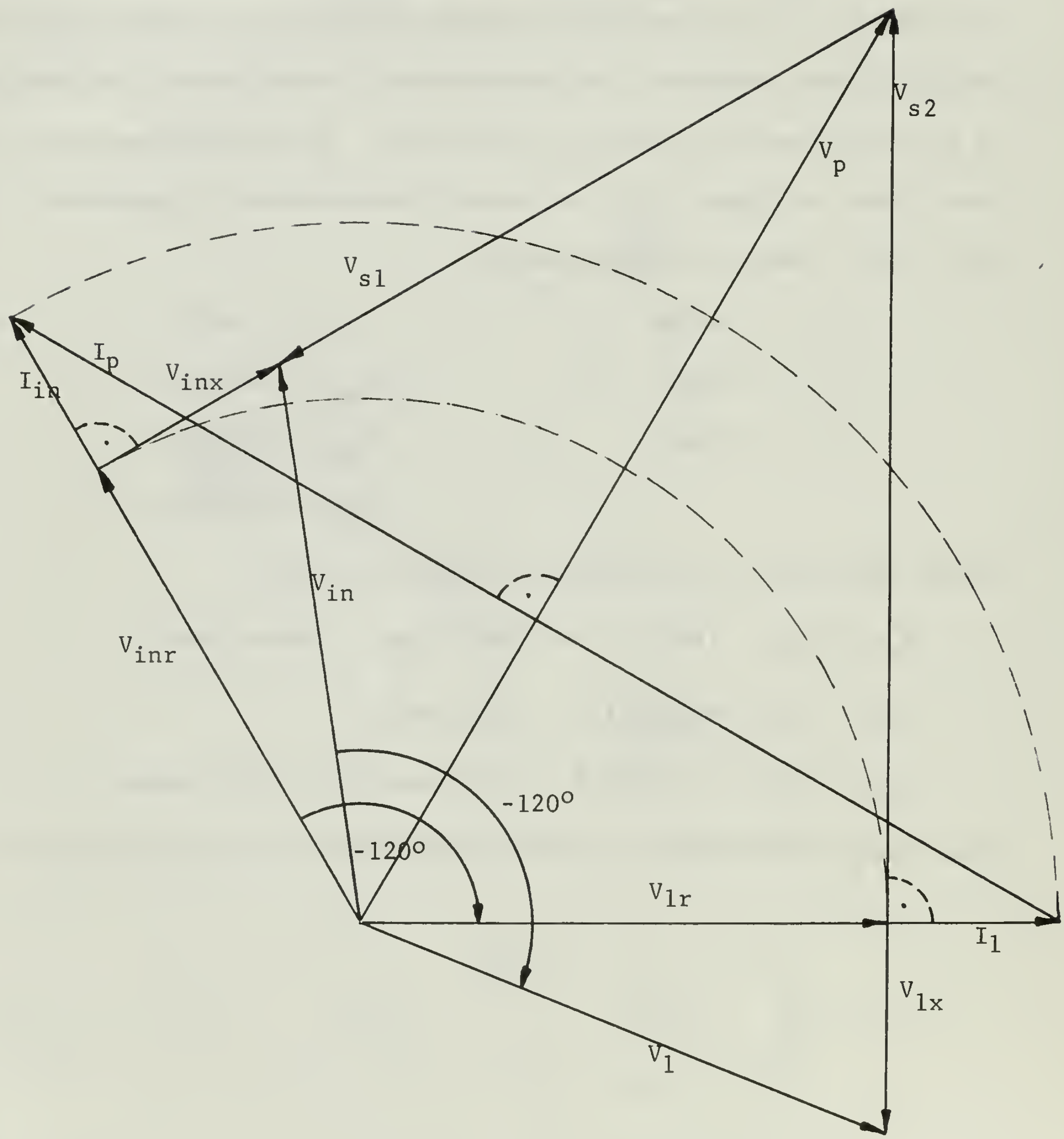


Figure 7-38. T-Network to Produce Desired Phase Shift Between Complex Z_1 and Complex Z_{in}, Actual Construction

CONCLUSIONS

In this paper graphical methods for electrical circuit analysis and synthesis have been treated that are applicable for one fixed frequency only. Necessarily this restricts the scope of this paper to a rather narrow field. If frequency is taken as another parameter, many more methods can be introduced, mostly related to the Smith Chart techniques. The literature contains more methods for variable frequency than for fixed frequency. It is interesting to note from Figure C-1 that the peak of publications in the field treated in this paper lies in the early forties. It was at that time that much work was devoted to the rising field of microwave techniques as well as to related waveguide problems. In these areas graphical methods are extremely useful and it seems that this has stimulated the development of graphical methods in other areas, too.

Many authors state that the advantages of graphical methods for analysis and synthesis of simple networks in electrical engineering are first, that they are simple; second, that they provide good insight; and third, that they are fast. In this paper a large number of different graphical methods has been discussed and it can be seen that only a few methods really show these advantages. Many methods are not simple at all. Their constructions are quite complicated. Often angles and distances have to be measured and transferred or special graph paper has to be used. In short, many of the described methods are for very special applications or of academic interest only.

The best application of graphical methods seems to be for engineers who perform the same type of analysis or synthesis problem over

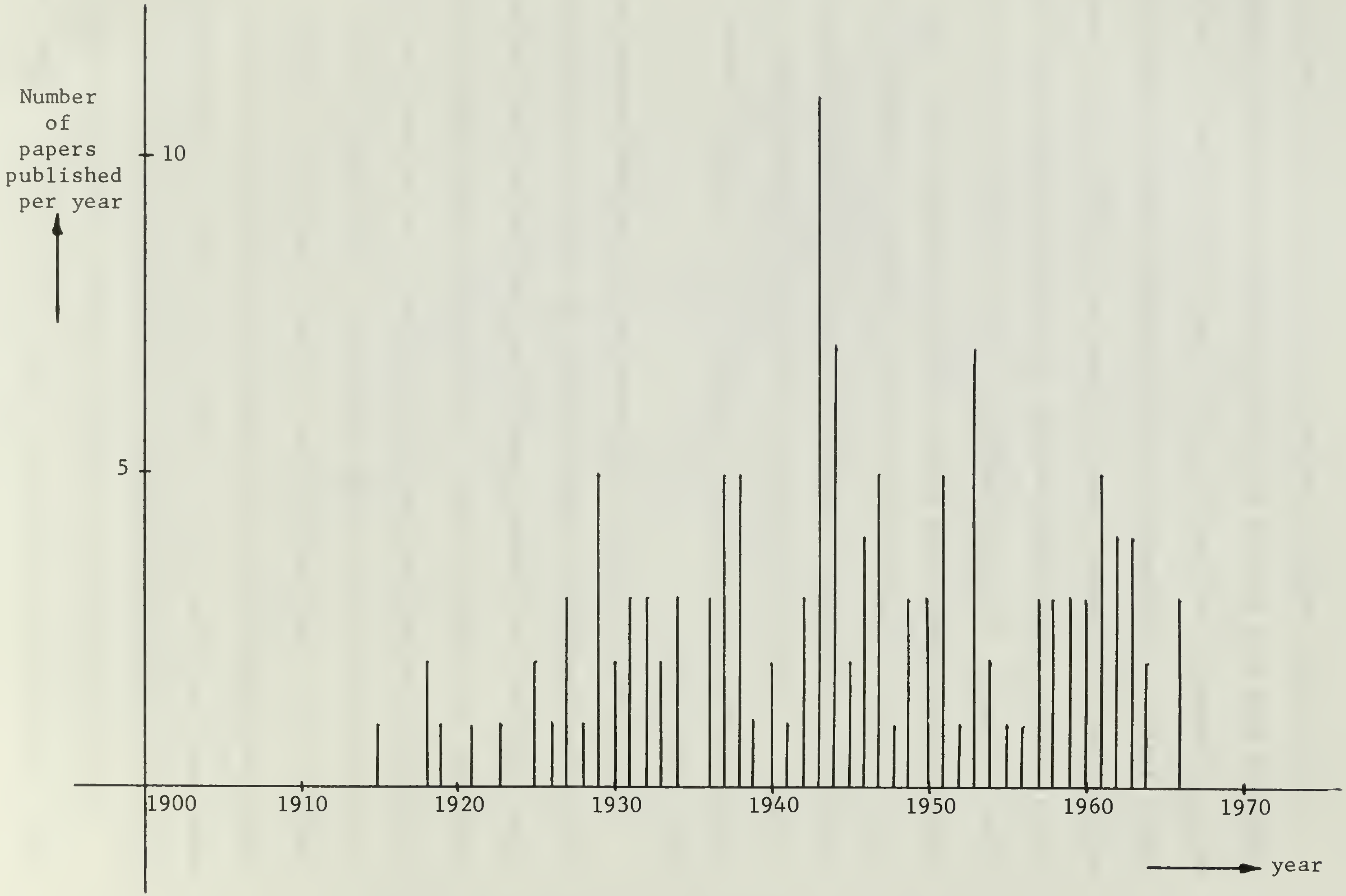


Figure C-1. Time Distribution of Publications

and over again. They select the method best suited for their problem and then use only this method. For someone who first has to learn the course of construction and then to apply it to an isolated problem it would seem that he could do the same job faster by resorting to analytical methods.

The indisputable merit of some methods lies in their educational value. In progressing step by step and clearly showing the result of each step, these methods provide good insight into the analysis or synthesis problem. Students in particular, can thus gain a good feeling for the physical nature of the problem, of the weight and importance of the different parameters, and their influence on the result.

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Appendix 1:

It is possible to show that the circuit in Figure 1-2 with R_s and X_s in series possesses the same input impedance as the one with R_p and X_p in parallel. In Figure 1-1 Z is given by

$$Z_p = \frac{jX_p R_p}{R_p + jX_p}$$

Multiplying numerator and denominator by the conjugate of the denominator yields:

$$Z_p = \frac{R_p X_p^2 + jX_p R_p^2}{R_p^2 + X_p^2}$$

But $R_p = \dfrac{Z}{\cos\theta}$ and $X_p = \dfrac{Z}{\sin\theta}$ which changes Z_p to

$$Z_p = \frac{Z\cos\theta + jZ\sin\theta}{\sin^2\theta + \cos^2\theta} = Z\cos\theta + jZ\sin\theta$$

Also from the same Figure

$$R_s = Z\cos\theta \qquad X_s = Z\sin\theta$$

Hence Z_p can equally well be written as

$$Z_p = R_s + jX_s = Z_s$$

and

$$\frac{jX_p R_p}{R_p + jX_p} = R_s + jX_s$$

Appendix 2:

From Figure 2-1 the similar triangles OBM, OCP and MAB, PAO can be taken. These give the following relationships:

MB/OP = OB/OC = AB/OA

But AB = OB - OA , which leads to OB/OC = (OB - OA) / OA.

X_p can be found by solving for OA.

$$OA = \frac{OB \cdot OC}{OB + OC} \quad \text{which is} \quad X_p = \frac{X_1 X_2}{X_1 + X_2} .$$

From Figure 2-2 the following relations can be taken:

$$\frac{MO + OP}{X_1} = \frac{MO}{X_p} \quad \text{and} \quad \frac{MO + OP}{X_2} = \frac{PO}{X_p} .$$

Solving for MO and equating the two equations yields

$$\frac{X_2 - X_p}{X_p} = \frac{X_p}{X_1 - X_p} ,$$ from which X_p can be easily isolated,

$$X_p = \frac{X_1 X_2}{X_1 + X_2} .$$

<u>Appendix 3</u>:

For the proof use is made of the similar triangles in Figure 2-7, BDA and BOC. They give the relations

$$BD / DA = BO / OC \quad \text{but} \quad BD = BO + OD \quad \text{and} \quad DA = OD$$

$$\frac{OD + BO}{OD} = \frac{BO}{OC} \quad \text{or} \quad 1 + \frac{BO}{OD} = \frac{BO}{OC} ,$$ which can be changed into

$$\frac{BO}{BO} + \frac{BO}{OD} = \frac{BO}{OC}$$ and by canceling BO this yields

$$\frac{1}{BO} + \frac{1}{OA} = \frac{1}{OC}$$ and solving for OC

$$OC = \frac{(BO)(OA)}{BO + OA} \quad \text{which is} \quad X_p = \frac{X_1 X_2}{X_1 + X_2} .$$

Appendix 4:

In Figure 2-10 triangles OCB and ODA are equilateral triangles. This makes BC parallel to OA and line OB parallel to AD.

The following proportions can be derived:

$$\frac{BC}{OA} = \frac{Z_2}{Z_1} \qquad \text{and} \qquad \frac{BC}{OA} = \frac{Z_2 - Z_p}{Z_p}$$

which yields when combined

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_{22}}$$

Appendix 5:

In Figure 3-1 distance OA is equal to distance DC. Then the following proportions can be derived:

$$ED : OA = OA : BD \quad \text{which results in} \quad (OA)^2 = (DF)(ED)$$

But OA represents in length Z_1 and DF represents in length $(Z_p - Z_1)$, whereas ED is equal to $(Z_2 - Z_1)$ resulting in

$$Z_p = \frac{|Z_1|\,|Z_2|}{|Z_2| - |Z_1|}$$

Since Z_1 is opposite in sign from Z_2, however,

$$Z_p = \frac{Z_1 \, Z_2}{Z_1 + Z_2}$$

Appendix 6:

In Figure 3-4 triangles ODC and EOB are similar,

hence

$$\frac{OB}{DC} = \frac{OE}{OD} = \frac{|Z_1|}{|Z_2|} .$$

The triangles OAB and DAC, too, are similar,

hence

$$\frac{OB}{DC} = \frac{OA}{DA} = \frac{OA}{OD + OA} = \frac{|Z_p|}{|Z_p| + |Z_2|} .$$

Setting the two above equations equal yields

$$\frac{|Z_1|}{|Z_2|} = \frac{|Z_p|}{|Z_p| + |Z_2|}$$ and solving for Z_p

$$|Z_p| = \frac{|Z_1|\,|Z_2|}{|Z_1| - |Z_2|}$$ and because Z_1 has the opposite sign of Z_2

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

Appendix 7:

In Figure 3-9 triangles OAM and B'OP form similar triangles. Then the following relationship is obtained:

$OA / OB' = CA / CO$ which represents

$$X_1 / X_c = (X_1 + X_p) / X_p$$

Solving this equation for X_p yields

$$X_p = \frac{X_1 X_c}{X_1 + X_c} .$$

Appendix 8:

From Figure 4-1 it can be seen that $DC / OA = BC / BA$

but $OA = X$, $BC = R$, and $BA = \sqrt{R^2 + X^2}$.

Hence $\dfrac{DC}{X} = \dfrac{R}{\sqrt{R^2 + X^2}}$ or $DC = |Z_p| = \dfrac{RX}{\sqrt{R^2 + X^2}}$

From Figure 4-2 the following relations can be found:

$$OD = \frac{OE\,|Z_p|^2}{X^2} \quad \text{and} \quad DE = \frac{OE\,|Z_p|^2}{R^2} ,$$

but $OE = OD + DE$ and therefore $OE = OD\,|Z_p|^2 \cdot (1/X^2 + 1/R^2)$ and

$1/|Z_p|^2 = 1/X^2 + 1/R^2$ which gives the desired relation

$$|Z_p|^2 = \frac{R^2 X^2}{R^2 + X^2} , \quad \text{hence} \quad |Z_p| = \frac{RX}{\sqrt{R^2 + X^2}} .$$

Figure 4-3 yields $OC/OB = OA/AB$ and $OC = \dfrac{(OB)(OA)}{AB}$, which

results in the desired $OC = Z_p = \dfrac{RX}{\sqrt{R^2 + X^2}}$.

Appendix 9:

In Figure 5-1 , OC is unity and triangles OAC and OBD are similar. It then follows $OA/OC = OB/OD$ and since $OC = OB = 1$ $OA = 1/OD$, from which the result is evident: $OA = 1/Z$, at the proper angle.

Appendix 10:

From Figure 5-3 the following similar triangles can be found:

PAK and MOK PAO and MOB

GML and OPL GMO and OPE

These triangles lead to the proportions:

$$GM / OP = GL / OL = OG / OE, \quad \text{where } GL = OG + OL$$

Hence

$$OG / OE = (OG + OL) / OL = OG / OL + OL / OL = OG / OL + OG / OG$$

Then

$$OG / OL = (OG / OE) - (OG / OG)$$

and

$$\frac{1}{OL} = \frac{1}{OE} - \frac{1}{OG}$$

OL is the parallel value of the imaginary components of the impedances Z_1 and Z_2.

The above similar triangles lead also to the relations:

$$PA / OM = KA / OK = OA / OB, \quad \text{where } KA = OA - OK$$

Hence

$$OA / OB = (OA - OK) / OK = OA / OK - OK / OK$$

Then

$$OA / OK = OA / OB + OA / OA$$

and

$$\frac{1}{OK} = \frac{1}{OB} + \frac{1}{OA}$$

OK is the parallel value of the real part of the impedances Z_1 and Z_2.

By series-parallel conversion of OK and OL Z_p is obtained.

Appendix 11:

From Figure 5-4 it can be seen that triangles ODB and OAC are similar triangles since all three angles are the same. This leads to the proportion

$$\frac{Z_2}{Z_1 + Z_2} = \frac{Z_p}{Z_1}$$ which in turn leads to

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Appendix 12:

In Figure 5-7 triangles ODH and OEG form similar triangles. This leads to

$$OD / OE = OH / OG \quad \text{and} \quad OH = \frac{(OD)(OG)}{OE}$$

Replacing OD, OG, and OE by the respective Z_2, Z_1, and $(Z_1 + Z_2)$ gives

$$OH = Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

Appendix 13:

For two impedances in parallel it must hold

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{Z_1 Z_2}{Z_s}$$

and from Figure 5-8 it can be seen that

$$|Z_p| = \frac{|Z_1|\,|Z_2|}{|Z_s|} = \frac{(OA)(OB)}{OC} .$$

Since OB = OE it follows that $|Z_p| = \frac{(OA)(OE)}{OC}$.

From similar triangles the relations can be taken

$$OA/OC = OF/OE \quad \text{and} \quad OF = \frac{(OA)(OE)}{OC} , \quad \text{hence} \quad OF = |Z_p| = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

For the argument of Z_p the following holds:

$$\begin{aligned} \mathrm{Arg}(Z_p) &= \mathrm{Arg}(Z_1) + \mathrm{Arg}(Z_2) - \mathrm{Arg}(Z_s) \\ &= \angle XOA + \angle XOB - \angle XOC = \angle XOD . \end{aligned}$$

Appendix 14:

This proof is originally given by Kreielsheimer and is based on the theory of vector loci. In Figure 5-9 Z_1, Z_2, and Z_p can be thought of as being vectors inverted about the origin. Then the original vectors have a magnitude of $\frac{1}{|Z_1|}$, $\frac{1}{|Z_2|}$, $\frac{1}{|Z_p|}$ respectively and phase angles which are opposite to those of Z_1, Z_2, and Z_p.

The vector $\frac{1}{|Z_p|} \angle -Z_p$ is then the sum $\frac{1}{Z_1} + \frac{1}{Z_2}$, that is

$$\frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

From this relation the final relation is obtained:

$$Z_p = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2},$$

the parallel value of Z_1 and Z_2.

Appendix 15:

In Figure 5-12 the secant theorem leads to the relationship $(OH)(OG) = (OD)(OA)$, and from the tangent theorem results the following $(OG)(OT) = (OT)(OH)$.

Now arbitrarily OT is set equal to 1.

Then $OG = 1/OH$ and $OA = 1/OD$.

Adding gives $OG + OA = OF = 1/Z_1 + 1/Z_2 = 1/Z_p$.

And since $OE = OB$ and $OC = OF$ the relation $OB = 1/OC$ yields

$$OE = \frac{1}{OF} = Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

Appendix 16:

The expression for two impedances in parallel is $Z_p = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$.

From Figure 5-13 it can be seen that

$Z_1 = OA$, $Z_2 = OF$, $Z_1 + Z_2 = OD$, and then $Z_p = \dfrac{(OA)(OF)}{OD}$.

Because $OA = OC$, $Z_p = \dfrac{(OD)(OF)}{OD}$.

From the ray theorem the relations can be found $OF/OD = OE/OC$

which gives $OE = \dfrac{(OC)(OF)}{OD} = Z_p$ and because $OE - OB$

$$OB = Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

Appendix 17:

In Figure 5-15 triangles OBD and OAD are similar since angles BAC and ODB are equal and each triangle possesses a right angle. Then

$$OB\ /\ OD = OD\ /\ OA \qquad \text{But} \quad OD = |Z_2| = 1, \quad \text{hence}$$

$$OB = \frac{OD^2}{OA} = \frac{1}{OA} \qquad \text{and} \qquad |Y_1| = \frac{|Z_2^2|}{|Z_1|} = \frac{1}{|Z_1|}$$

Triangles OHA and OBG are similar since angles OGB and OAH are equal and angle BOH is common to both. Then

$$OA\ /\ OH = OG\ /\ OB \qquad \text{and} \qquad OG = \frac{1}{OH}\,(OA)\cdot(OB)$$

But $OB = 1/OA$, hence $OG = 1/OH$

Vector addition leads to $OG = OB + OF = Z_2^2/Z_1 + Z_2 = (Z_2^2 + Z_1Z_2)/Z_1$

Inverting this and using a multiplication factor of Z_2^2 (which is equal to one because it is normalized) the following results are obtained:

$$1/OG = (Z_1/(Z_2^2 + Z_1Z_2)) \times Z_2^2 \qquad \text{which is} \qquad \frac{1}{OG} = \frac{Z_1\ Z_2}{Z_1 + Z_2}$$

But since $\frac{1}{OG} = OH$,

then

$$OH \doteq Z_p = \frac{Z_1\ Z_2}{Z_1 + Z_2}$$

Appendix 18:

For two impedances Z_1 and Z_2 in parallel the resultant impedance can be written as

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{Z_1}{\frac{Z_1}{Z_2} + 1} .$$

This equation can be considered as a linear fractional transformation in Z_1 or Z_2. It can be written as $Z_p - \frac{aZ_1 + b}{cZ_1 + d}$ and for reciprocal networks $ad - bc = 1$, from which $Z_p = \frac{Z_1}{\frac{Z_1}{Z_2} + 1}$,

if $a = 1$, $b = 0$, $c = \frac{1}{Z_2}$ and $d = 1$.

In Figure 5-16, the isometric circle for direct transformation, b , has its center at the tip of $OB' = -\frac{d}{c} = -Z_2$ and its radius is

$$r_b = \frac{1}{|c|} = |Z_2| .$$

The isometric circle for inverse transformation, c , has its center at the tip of $OB = \frac{a}{c} = Z_2$ and its radius is $r_c = \frac{1}{|c|} = |Z_2|$.

Appendix 19:

In Figure 5-21 triangles OAD and OBC are congruent because all angles and sides are equal. Hence

$$\frac{AO}{AD} = \frac{OC}{BC} = \frac{Z_2}{Z_1 + Z_2} \quad .$$

Triangles OBC and OFE are similar because all angles are equal. Hence

$$\frac{AO}{AD} = \frac{OF}{OE} = \frac{Z_p}{Z_1}$$

Equating the above two equations yields

$$\frac{Z_2}{Z_1 + Z_2} = \frac{Z_p}{Z_1}$$

which can be combined to give

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Appendix 20:

In Figure 5-22 it is shown that

$$OA + AB = OB = Z_s \qquad \text{and angle } AOB = \theta_s$$

From similar triangles ODE and OBC the following proportions are obtained:

$OE / OC = OD / OB$ or replacing OC by Z_2 and OD by Z_1, OB by Z_s

$OE / |Z_2| = |Z_1| / |Z_s|$. This results in $OE = Z_p = \dfrac{|Z_1|\,|Z_2|}{|Z_s|}$

As for the angles:

$$\theta_p = (90 - \theta_s) - (90 - \theta) \quad \text{or}$$

$$\theta_p = \theta - \theta_s \ ,$$

If an auxiliary triangle were formed, erecting a perpendicular on the x-axis going through B.

Appendix 21:

Triangles AOC and ODC in Figure 5-23 are similar. By the secant tangent theorem

$$OD / OC = OA / AC \quad \text{leading to} \quad OD = \frac{(OA)(OC)}{AC}$$

This however is the same as $OD = \dfrac{|Z| \cdot |R|}{|Z| + |R|} = |X_p|$

As to the angle $\angle DOC = \angle BOC - \angle OCA$, since triangles BDO and AOC are similar, too, $\angle BOD = \angle OCA$. Then $\angle DOC = \angle BOC - \angle BOD$

Appendix 22:

From Figure 5-24 the following relations can be derived:

OH/OD = FG/FD which is $\frac{Z_p}{Z_2} = \frac{Z_1}{Z_1 + Z_2}$ and $Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$.

From Figure 5-24 it is possible to derive the equations for the equivalent resistance and reactance values of Z_p. One gets

$$R_p = Z_p^2 \cdot \left(\frac{R_1}{Z_1^2} + \frac{R_2}{Z_2^2} \right) \quad \text{and} \quad X_p = Z_p^2 \cdot \left(\frac{X_1}{Z_1^2} + \frac{X_2}{Z_2^2} \right) .$$

If $\mathrm{Arg}(Z_p) = \theta_p$ from the above two equations an expression for θ_p can be found:

$$\cos \theta_p = \frac{R_p}{X_p} = \frac{1}{Z_1 + Z_2} \cdot (Z_2 \cos \theta_1 + Z_1 \cos \theta_2)$$

or $$(Z_1 + Z_2) \cdot \cos \theta_p = Z_2 \cdot \cos \theta_1 + Z_1 \cdot \cos \theta_2 .$$

Figure 5-25 gives now $OB'' = Z_2 \cos \theta_1$ and $OA'' = Z_1 \cos \theta_2$

and $OC'' = OA'' + OB'' = Z_2 \cos \theta_1 + Z_1 \cos \theta_2$

and it can be seen that

$$OC'' = (OJ) \cos \theta_p = (Z_1 + Z_2) \text{ Cos } \theta_p .$$

Therefore

$$(Z_1 + Z_2) \cos \theta_p = Z_2 \cos \theta_1 + Z_1 \cos \theta_2 ,$$

and

$$OH' = |Z_p| \cdot \angle \theta_p .$$

Appendix 23:

The similar triangles OCB and OAD in Figure 5-26 yield the relation

$$OC \,/\, OB = OA \,/\, OD$$

It follows therefrom that

$$OC = (OB) \cdot \frac{(OA)}{OD} \quad \text{or} \quad OC = \frac{Z_1 \cdot Z_2}{Z_2 + Z_1} = Z_p$$

which is the final result.

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13. ABSTRACT

This thesis, the result of a literature search conducted at the Naval Postgraduate School, is a collection of different geometrical methods and their applications in respect to network analysis and synthesis in electrical engineering.

Part one deals primarily with finding the equivalent of two or more impedances in parallel. The second part is directed towards the application of these methods to the problem of matching two arbitrary impedances in order to obtain maximum power transfer.

A comprehensive bibliography is included.

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